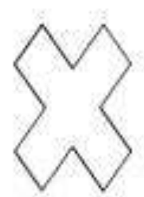
<u>Modern Algebra I: Group Theory</u>

Class log for Tuesday, January 8th

Task 1
Introduction/Syllabus

Task 2
Symmetry can be thought of in two different ways:
1. A symmetry is a property of a figure.
2. A symmetry is an isometry (rigid motion) that maps a figure to itself.
   (An isometry is a transformation that preserves angles and distances.)

Task 3
Consider the following figure:



1. How many symmetries does this figure have?
2. Come to a consensus and describe each symmetry.

Task 4
Definition: Two symmetries of a figure are equivalent if they have the same effect on the figure.

Task 5
Determine how many symmetries an equilateral triangle has. For each symmetry:
1. Write a verbal description of the symmetry.
2. Draw a diagram to illustrate the symmetry.

Task 6
How do we know there are only six symmetries? Justify that there can be no more.

Task 7
What happens if you combine two of the six symmetries? Is the combination a new symmetry or equivalent to one of the original six? Justify!

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Class log for Tuesday, January 15th

Task 1
Let F stand for a flip across the vertical axis and R stand for a 120º clockwise rotation. For each of your six symmetries, find a combination of F's and R's that is equivalent.

Task 2
Select a set of symbols (in terms of F and R) to use as a class. Fill out the table in the given handout for the new symbols so that we have a nice record of what each symmetry does to the triangle.

Task 3
Earlier we decided that any combination of two symmetries had to be equivalent to one of our six symmetries. Your task: For each combination of two symmetries, figure out which of our six it is equivalent to. Display the results in an operation table.

Task 4
Did you figure all of them out by moving the triangle around or did you use some shortcuts (“rules”) to do the calculations? Make a list of these rules.

Task 5
Were any of the rules more useful than others? Pick the rule that you used the most often, and see if you can prove the other rules from it.

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Class log for Thursday, January 17th

Task 1
Prove the "Sudoku" property for the distinct symmetries of an equilateral triangle. Do this by first proving that each element appears *no more than once* in a row/column, and then by proving that each element occurs *at least once*.

Task 2
Which axioms did you make use of to prove the Sudoku property (specifically, the cancellation law)?

Task 3
Do these axioms (identity, associativity, and inverse) hold for the symmetries of a non-square rectangle? Do they hold for any complete set of (distinct) symmetries? Explain.

Task 4
The three primary axioms you listed are the axioms for an algebraic structure called a *group* – a structure around which much of abstract algebra (and this course) is based (we will be formalizing this definition very soon). Can you come up with other, perhaps more familiar, examples of structures that satisfy these axioms?

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Class log for Tuesday, January 22nd

Task 1
Recall (from the homework) the set of symmetries of a non-square rectangle and the set of symmetries of the rotations of a square. Construct their operation tables here. Are these two sets of symmetries essentially the same? Why or why not? Identify any similarities or differences that you notice between the two.

Task 2
Can a group have two (distinct) identity elements?
Can a group element have two (distinct) inverses?

Task 3
Prove or disprove that these are groups:

The set {-1, 0, 1} with regular addition
The set {-1, 0, 1} with regular multiplication
The set {-1, 1} with multiplication

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Class log for Thursday, January 24th

Task 1 (continuation of Task 4 on Thursday, 1/17)

The three primary axioms you listed are the axioms for an algebraic structure called a *group* – a structure around which much of abstract algebra (and this course) is based. Can you come up with other, perhaps more familiar, examples of structures that satisfy these axioms? For example, are the integers with addition a group?

Task 2

Write a formal mathematical definition to complete the sentence: A *group* is …

Task 3

The terms *abelian* (commutative) *group* and *subgroup* were defined. We discussed future plans to discern a minimal set of conditions for a subset to be a subgroup and to find all subgroups of a given group.

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Class log for Tuesday, January 29th

Task 1

As a class, complete the following true/false activity involving groups and subgroups:

Circle True or False. If false, give a counter-example or state what is required to make it true. 3 points each.

1. True or False: The identity element in a group G is its own inverse.

2. True or False: If G is an abelian group, then $x^{-1} = x$ for all x in G.

3. True or False: Let G be a group that is not abelian then $xy \neq yx$ for some x, y in G.

4. True or False: Let x, y, z be elements of a group G. Then $(xyz)^{-1} = z^{-1}y^{-1}x^{-1}$.

5. True or False: Every group G contains at least two subgroups.

6. True or False: The identity element in a subgroup H of a group G must be the same as the identity element in G.

7. True or False: If a subgroup H of a group G is abelian, then G must be abelian.

8. True or False: Every subgroup of an abelian group is abelian.

Task 2

We introduced the groups Z_n under addition modulo *n*, but noted that things were not so straightforward with multiplication modulo *n*. The question was then posed: under what conditions is {1, …, *n*-1} with multiplication modulo *n* a group? (Hint: try a few smaller cases out with operation tables!) Can you prove your conjecture?

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Class log for Thursday, January 31st

Task 1

Given an element a of a group G, we define the order of a to be the smallest positive integer k such that $a^k = e$, where e is the identity element of G.

Find the order of each element of D_4 (the symmetries of a square).

Task 2

Let G be a group. Suppose that $a,b \in G$, $o(a) = 2$, $o(b) = 4$ and $ab = b^3a$.

a. Prove that $bab = a$.
b. Prove that $ab^2 = b^2a$.

Task 3

Two "if and only if" proofs:

a. (a) Let G be a group. Show that G is commutative if and only if $(ab)^2 = a^2b^2$ *for all* $a, b \in G$.
b. (b) Let G be a group. Show that G is commutative if and only if $(ab)^{-1} = a^{-1} b^{-1}$ for all $a, b \in G$.

Task 4

In general what is $(ab)^{-1}$? Prove your assertion.

Task 5

Prove or disprove that each of the tables below are a group.

+	0	1
0		
1		

×	0	1
0		
1		

*	a	b
a	b	a
b	a	b

+	EVEN	ODD
EVEN		
ODD		

Modern Algebra I: Group Theory

Class log for Tuesday, February 5th

Task 1

Definition: A subgroup is a subset of a group that is itself a group with respect to the same operation.
Prove that $(5\boldsymbol{Z},+)$ is a subgroup of $(\boldsymbol{Z}, +)$:

1. *Closure*: Let $a,b \in 5\boldsymbol{Z}$.
   …
   Therefore $a+b \in 5\boldsymbol{Z}$.

2. *Identity:* 0 is the identity of $\boldsymbol{Z}$.
   …
   Therefore $0 \in 5\boldsymbol{Z}$.

3. *Inverses:* Let $a \in 5\boldsymbol{Z}$.
   …
   Therefore $a^{-1} \in 5\boldsymbol{Z}$.

4. *Associativity*
   $5\boldsymbol{Z}$ inherits associativity from $\boldsymbol{Z}$ (you always get this one for free when proving that something is a subgroup, provided that it is a subset!).

Task 2

Time to make a theorem!

According to our definition of subgroup, we must check all four group axioms to ensure a subset is a group (Closure, Identity, Inverses, Associativity).

Come up with a smaller set of conditions that are sufficient (and necessary) to ensure that a subset of a group is a subgroup.

Note: What we want is fewer conditions, so that it will be easier to prove things are subgroups. There is more than one such set of conditions.

Write and prove a theorem based on your conditions.

Task 3

Suppose that $(G, *)$ is a group, and suppose that H is a subgroup of G.

1. Prove that the identity of H must be the same as the identity of G.

2. Prove that the inverse of any element of H must be the same as that element's inverse in G

Modern Algebra I: Group Theory

Class log for Thursday, February 7th

In today's class, we reviewed the proof of the subgroup test (Theorem # 10), and began to document all examples, definitions, and theorems from this course to date.

<u>Modern Algebra I: Group Theory</u>

Class log for Tuesday, February 12th

Task 1
Consider this conjecture:

"Any closed subset of a group is a subgroup."

Prove or disprove this conjecture.

Task 2
Consider the following "proof" of the conjecture that closure is sufficient:

"Because the subset is closed, only elements from the subset appear in a row of the table. Because the entire group table has the Sudoku property, any closed subset does as well. So each element of the subset must appear in any row. Since an element *a* must appear in the row for symmetry *a*, the identity must be in the subset. Then since the identity must appear in the row for *a*, the inverse of *a* must be in the subset. Finally, since associativity holds for the whole group it must hold for the subset. Therefore the subset is a group and hence a subgroup."

Where is the flaw in this argument? Are there cases where this proof works? Can this be used to make a new subgroup theorem?

Task 3
Could this be an operation table for the symmetries of an equilateral triangle?

The *Mystery* Group

*	**A**	**B**	**C**	**D**	**E**	**G**
A	B	A	D	C	G	E
B	A	B	C	D	E	G
C	G	C	B	E	D	A
D	E	D	A	G	C	B
E	D	E	G	A	B	C
G	C	G	E	B	A	D

Task 4
Why doesn't the correspondence

$$\text{A} \leftrightarrow F,\ \text{B} \leftrightarrow \text{I},\ \text{C} \leftrightarrow FR^2,\ \text{D} \leftrightarrow R,\ \text{E} \leftrightarrow FR,\ \text{G} \leftrightarrow R^2$$

work for showing the mystery group is really $\boldsymbol{D}_3$?