Modern Algebra I: Class Activities

Thursday, February 14th

Task 1

Definition: Let $(G, \bullet)$ and $(H,*)$ be groups. G is *isomorphic* to H if there exists a bijective function $\varphi: G \to H$ such that $\varphi(a \bullet b) = \varphi(a) * \varphi(b)$ for every $a, b \in G$. The function φ is called an *isomorphism*.

Task 2

Consider the group whose operation is given by the table below.

*	**A**	**B**	**C**	**D**
A	D	C	B	A
B	C	A	D	B
C	B	D	A	C
D	A	B	C	D

a. Is it isomorphic to C_4, the rotations of a square? Justify using the definition.

b. Is it isomorphic to V_4, the symmetries of a non-square rectangle? Justify using the definition.

Task 3

Now let's prove some theorems that are obvious given our informal idea that isomorphic groups are essentially the same (but must be proven now using our formal definition).

a. Theorem: Suppose $(G, \bullet)$ and $(H, *)$ are isomorphic groups. Then if G is abelian then H is abelian.

b. Theorem: Suppose $(G, \bullet)$ and $(H, *)$ are groups (with identities e_G and e_H, respectively) $\varphi: G \rightarrow H$ is an isomorphism. Then $\varphi(e_G) = \varphi(e_H)$.

c. Theorem: Suppose $(G, \bullet)$ and $(H, *)$ are groups and $\varphi: G \rightarrow H$ is an isomorphism. Let $a \in G$, then $\varphi(a)^{-1} = \varphi(a^{-1})$.

Task 4

We need to prove a little lemma so we can prove some more cool (and obvious) theorems.

Lemma: Suppose $(G, \bullet)$ and $(H,*)$ are groups and $\varphi: G \to H$ is an isomorphism. Let $a \in G$ and $n \in \mathbb{N}$, then $\varphi(a)^n = \varphi(a^n)$.

This one is a "routine" proof by mathematical induction. We want to show it is true for any integer. We do this in two steps.

First: Prove it works for $n = 1$.

Second: Prove that if it works for $n = k$, it works for $n = k+1$. (Hint: If you get stuck, work it out for $n = 2$ to get an idea).

Together these imply it works for any $n \in \mathbb{N}$.

Task 5

A theorem about Isomorphic Groups: Suppose G and H are isomorphic groups then, if G is cyclic, then H is cyclic.

Task 6

In this theorem, we are going to prove that isomorphisms preserve the order of an element: Suppose $(G, \bullet)$ and $(H,*)$ are groups and $\varphi: G \to H$ is an isomorphism. Let $a \in G$. Then $|\varphi(a)| = |a|$.