Mathematics 2243: Calculus III Solutions to Practice Exam 2

Name (please print) ______________________

Student Number ______________

I. (15) For each of the following power series, use one of the following tests to determine convergence or divergence: Integral Test, Comparison Test, Alternating Series Test, Ratio Test.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$f(x) = \frac{1}{\sqrt{x}}$ is decreasing and positive

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{M\to\infty} \int_1^{M} \frac{1}{\sqrt{x}}\,dx = \lim_{M\to\infty} 2\sqrt{x}\Big|_1^M$$

$$= \lim_{M\to\infty} 2\sqrt{M} - 2 = \infty$$ so the series diverges by the Integral Test

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

This is $\Sigma(-1)^n b_n$ where $b_n = \frac{1}{\sqrt{n}}$.

b_n is decreasing and $\lim_{n\to\infty} b_n = 0$.

So the series converges by the Alternating Series Test

3. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2}$

$$0 \le \frac{\sin^2(n)}{n^2} \le \frac{1}{n^2}.$$

Since $\Sigma \frac{1}{n^2}$ converges, the series converges by the Comparison Test.

Mathematics 2243: Calculus III Solutions to Practice Exam 2

II. (15) For each of the following, circle the letter of the correct response.

1. For which values of p does the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge? *p-series*

(a) all values (b) $p < 0$ (c) $p \leq 0$

(d) $p < 1$ (e) $p \leq 1$ (f) $p > 0$

(f) $p \geq 0$ (g) $p > 1$ (i) $p \geq 1$

2. To show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+2}$ converges, one could compare its terms to the terms of which of the following series?

~~(a)~~ $\sum_{n=1}^{\infty} \frac{1}{n}$ *divergent*

~~(b)~~ $\sum_{n=1}^{\infty} \frac{1}{2^n}$ *terms smaller than* $\frac{1}{n^2+2n+2}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^2}{n^2}$

~~(d)~~ $\sum_{n=1}^{\infty} \arctan(n+1)$ *divergent*

~~(e)~~ $\sum_{n=1}^{\infty} \ln(n^2+2n+2)$ *divergent*

~~(f)~~ $\sum_{n=1}^{\infty} \frac{1}{n^3}$ *terms smaller*

~~(g)~~ $\sum_{n=1}^{\infty} \frac{1}{n^4+n^3+2}$ *terms smaller than* $\frac{1}{n^2+2n+2}$

~~(h)~~ more than one of these

(i) none of these

3. The series $\sum_{n=0}^{\infty} 2^n x^{2n}$ converges for

(a) only $x = 0$ (b) $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ (c) $-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$

(d) $-\frac{1}{2} < x < \frac{1}{2}$ (e) $-\frac{1}{2} \leq x \leq \frac{1}{2}$ (f) $-1 < x < 1$

(g) $-1 \leq x \leq 1$ (h) $-\sqrt{2} < x < \sqrt{2}$ (i) $-\sqrt{2} \leq x \leq \sqrt{2}$

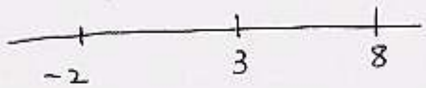
(j) $-2 < x < 2$ (k) $-2 \leq x \leq 2$ (l) $-\infty < x < \infty$

$\sum_{n=0}^{\infty} (2x^2)^n$ *geometric. Converges when* $|2x^2| < 1$

$$2x^2 < 1$$
$$x^2 < \frac{1}{2}$$
$$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

Mathematics 2243: Calculus III Solutions to Practice Exam 2

III. (5) A power series of the form $\sum_{n=0}^{\infty} c_n(x-3)^n$ has radius of convergence $R = 5$. What can be said about its convergence or divergence at different values of x?



it converges absolutely for $-2 < x < 8$

it diverges for $x < -2$ or $x > 8$

at each of $x = -2$ and $x = 8$, it might diverge or converge conditionally or converge absolutely

IV. (7) The Maclaurin series for $\sin(x)$ is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

(a) Use the ratio test to verify that this series converges absolutely for all x.

$$\lim_{n\to\infty} \frac{|x|^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{|x|^{2n+1}} = \lim_{n\to\infty} \frac{|x|^2}{(2n+2)(2n+3)} = 0$$

(b) Differentiate this series term by term to obtain a power series for $\cos(x)$.

$$\sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)\, x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Mathematics 2243: Calculus III Solutions to Practice Exam 2

V. (8) Starting with the fact that

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \frac{1}{1-(-x)}\,dx\,,$$

expand $\frac{1}{1-(-x)}$ as a power series, and integrate to obtain a power series, plus a constant, which is equal to $\ln(1+x)$. Evaluate at $x=0$ to show that the constant equals 0.

$$= \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$$

For $x=0$, $\ln(1+0) = C + 0$ so $C = 0$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

VI. (6) Define *absolutely convergent* and *conditionally convergent*. Give an example of a conditionally convergent series.

$\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.

$\sum a_n$ is conditionally convergent if $\sum a_n$ converges and $\sum |a_n|$ diverges, as for example the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

(it converges by the Alternating Series Test, but $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, which diverges)