Calculus III: Homework Solutions

Answers to odd-numbered textbook exercises can be found in the back of the textbook (and consequently are not included here).

Section 11.3

Solutions to Even-Numbered Textbook Exercises:

6. converges; $\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}} = -2\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which is a convergent p-series ($p = \frac{3}{2}$)

14. diverges by the Integral Test: $\int_1^n \frac{dx}{2x-1} = \frac{1}{2}\ln(2n-1) \rightarrow \infty$ as $n \rightarrow \infty$

Section 11.4

2. diverges by the Direct Comparison Test since $n + n + n > n + \sqrt{n} + 0 \Rightarrow \frac{3}{n+\sqrt{n}} > \frac{1}{n}$, which is the nth term of the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$ or use Limit Comparison Test with $b_n = \frac{1}{n}$

6. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^{3/2}}$, the nth term of a convergent p-series: $\lim_{n \to \infty} \frac{\left(\frac{n+1}{n^2\sqrt{n}}\right)}{\left(\frac{1}{n^{3/2}}\right)} = \lim_{n \to \infty} \left(\frac{n+1}{n}\right) = 1$

Section 11.5

2. converges by the Ratio Test: $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\left(\frac{(n+1)^2}{e^{n+1}}\right)}{\left(\frac{n^2}{e^n}\right)} = \lim_{n \to \infty} \frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^2 \left(\frac{1}{e}\right) = \frac{1}{e} < 1$

4. diverges by the Ratio Test: $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\left(\frac{(n+1)!}{10^{n+1}}\right)}{\left(\frac{n!}{10^n}\right)} = \lim_{n \to \infty} \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} = \lim_{n \to \infty} \frac{n}{10} = \infty$

16. converges by the Ratio Test: $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)\ln(n+1)}{2^{n+1}} \cdot \frac{2^n}{n\ln(n)} = \frac{1}{2} < 1$

22. converges by the Ratio Test: $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n \to \infty} \left(\frac{n}{n+1}\right)^n = \lim_{n \to \infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}$
$= \lim_{n \to \infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$

Section 11.6

14. converges conditionally since $\frac{1}{1+\sqrt{n}} > \frac{1}{1+\sqrt{n+1}} > 0$ and $\lim_{n \to \infty} \frac{1}{1+\sqrt{n}} = 0 \Rightarrow$ convergence; but $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ is a divergent series since $\frac{1}{1+\sqrt{n}} \geq \frac{1}{2\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a divergent p-series

18. converges absolutely because the series $\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|$ converges by the Direct Comparison Test since $\left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}$

26. converges conditionally since $f(x) = \frac{1}{x \ln x} \Rightarrow f'(x) = -\frac{[\ln(x) + 1]}{(x \ln x)^2} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow u_n > u_{n+1} > 0$ for $n \geq 2$ and $\lim\limits_{n \to \infty} \frac{1}{n \ln n} = 0 \Rightarrow$ convergence; but by the Integral Test,

$$\int_2^\infty \frac{dx}{x \ln x} = \lim_{b \to \infty} \int_2^b \left(\frac{\left(\frac{1}{x}\right)}{\ln x} \right) dx = \lim_{b \to \infty} [\ln(\ln x)]_2^b = \lim_{b \to \infty} [\ln(\ln b) - \ln(\ln 2)] = \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n \ln n} \text{ diverges}$$

42. converges conditionally since $\left\{ \frac{1}{\sqrt{n} + \sqrt{n+1}} \right\}$ is a decreasing sequence of positive terms converging to 0 $\Rightarrow \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n} + \sqrt{n+1}}$ converges; but $\lim\limits_{n \to \infty} \frac{\left(\frac{1}{\sqrt{n} + \sqrt{n+1}}\right)}{\left(\frac{1}{\sqrt{n}}\right)} = \lim\limits_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n} + \sqrt{n+1}} = \lim\limits_{n \to \infty} \frac{1}{1 + \sqrt{1 + \frac{1}{n}}} = \frac{1}{2}$ so that $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n} + \sqrt{n+1}}$ diverges by the Limit Comparison Test with $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a divergent p-series