<u>Modern Algebra I: Group Theory Homework 6</u>

due Thursday, February 14th

19. Consider the 2x2 general linear group with entries from $\mathbb{Z}_2$, i.e. $GL(2, \mathbb{Z}_2)$.
    a. List all the elements of $GL(2, \mathbb{Z}_2)$. Verify that their determinants are nonzero.
    b. Construct an operation table.
    c. Is this group the same as D_3? Give a specific argument (that is, state which elements "correspond" to each other and how you know this to be the case).

20. Let $\mathbb{Z}_n^{\times}$ be defined as the set of elements in $\mathbb{Z}_n$ that have a multiplicative inverse modulo *n*. Formally, this is called the *set of units modulo n*. In general, putting the 'cross' symbol next to a set narrows the focus to all of the elements with multiplicative inverses. For example, the units in the set of 2x2 matrices comprise the general linear group, i.e. $M_2(\mathbb{R})^{\times} = GL(2, \mathbb{R})$.
    a. List all of the elements of $\mathbb{Z}_4^{\times}, \mathbb{Z}_5^{\times}, \mathbb{Z}_{12}^{\times}$.
    b. Construct operation tables for each of $\mathbb{Z}_4^{\times}$, $\mathbb{Z}_5^{\times}$, and $\mathbb{Z}_{12}^{\times}$ under multiplication modulo 4, 5, and 12, respectively. Are these examples of groups? Justify your assertions. (You may assume that the modular operations are associative.)
    c. Briefly explain why, for example, $\mathbb{Z}_{12}$ is not a group under multiplication modulo 12, but $\mathbb{Z}_{12}^{\times}$ is.

21. Find the order of each element in the following groups:
    a. D_3
    b. D_4
    c. V_4
    d. $\mathbb{Z}_4^{\times}$
    e. $\mathbb{Z}_5^{\times}$
    f. $\mathbb{Z}_{12}^{\times}$
    g. $GL(2, \mathbb{Z}_2)$
    h. Do you notice anything about the orders of the elements and the order of the group (the number of elements in the group)?

22. **Definition**: A *cyclic group* is a group that is generated by one element. That is, a cyclic group is comprised completely of powers of one element (as in the real numbers, we consider the identity to be the 0th power). For example, an arbitrary finite cyclic group *G* looks like *G*= $\{e, g, g^2, g^3, g^4, \ldots, g^n\}$ (an infinite cyclic group would not end at g^n but would keep going). For an additive group, this would be *G*= $\{e, g, 2g, \ldots, ng\}$. The shorthand for "*G* is generated by *g*" is $G = \langle g \rangle$.
    a. Prove that the finite group *G* above is indeed a group.
    b. Prove that the cyclic subset of D_4 generated by *R* (i.e. only the planar rotations) is a subgroup of D_4 by arguing that it is closed (i.e. by using the subgroup of a finite group theorem).
    c. Prove that any cyclic subset of a finite group *G* is an *abelian* subgroup by arguing that it is closed and commutative. (Start with a group *G* and a cyclic subset $\{e, a, a^2, a^3, \ldots, a^n\}$ of *G*, then prove that it is closed and commutative.)

23. It is a common task to find *all* subgroups of a given group. While there is unfortunately no set way to do this, some good places to start are: the centralizers of the elements (which we

noticed were all subgroups of D_4 on a previous homework) the center, and the cyclic subgroups. (Note that this will not necessarily give a *complete* list.)

a. List all of the subgroups of D_3.
b. Use what we know about D_3 and $GL(2, \mathbb{Z}_2)$ being "the same" to find all the subgroups of $GL(2, \mathbb{Z}_2)$. *Note:* you should not have to do any matrix calculations.
c. List all of the subgroups of D_4 (the group of symmetries of a square).
d. Do you notice a pattern between the order of the subgroups and the order of the ambient group (finite case only)?
e. List all of the subgroups of $(\mathbb{Z}, +)$.

Also included in this assignment (with the same due date): fill out the worksheets recording our examples of groups, definitions, and theorems (with proofs). Remember, these worksheets comprise our "textbook", so it is extremely important to get everything precisely correct (just as you would expect it to be in a textbook!). If you would like for me to look over and check your work before making an entry, just let me know!