Modern Algebra I: Group Theory Homework 5

due Thursday, February 7^{th}

18. Let $M_2(R)$ denote the set of 2x2 matrices with real entries.

a. Recall that matrix addition is defined component-wise as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} a+x & b+y \\ c+z & d+w \end{bmatrix}$$

Prove or disprove that $M_2(R)$ is a group under (matrix) addition.

b. Recall that matrix multiplication is defined as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{bmatrix}.$$

i. Identify the identity matrix I under this operation and show that $AI=A=IA$.

ii. Verify that the multiplicative inverse of the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $M^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ by showing that $MM^{-1} = I = M^{-1}M$.

iii. Prove that matrix multiplication is associative.

iv. Prove or disprove that $M_2(R)$ is a group under (matrix) multiplication.

c. The *determinant* of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, usually denoted $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, is given by *ad-bc*.

i. Compute the determinants of the following matrices: $A = \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -1 \\ 4 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}$.

ii. Notice that the determinant of a matrix is the denominator of the constant in its inverse formula. What relationship can you deduce between multiplicative inverses and the determinant? Explain.

iii. Find the products AB, AC, BC, AB^2, A^2B. Find the determinants of these matrices. Do you see a relationship between the determinants of A, B, and C and the determinants of their products (think: is there a way to compute these determinants without computing their products?)? State and prove your conjecture.

d. The set of 2x2 real matrices of nonzero determinant is called the *general linear group*, and is denoted $GL_2(R)$. Formally, $GL_2(R) = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R) : ad-bc \neq 0\right\}$. In this problem, we will prove that the general linear group is indeed a group under matrix multiplication.

i. Using your results from part (c-iii), prove that matrix multiplication is a binary operation on $GL_2(R)$ (that is, prove that $GL_2(R)$ is closed under matrix multiplication: if A and B are elements of $GL_2(R)$, then so is AB …).

ii. Using part (b) above, conclude that $GL_2(R)$ is a group.

e. The set of 2x2 real matrices of determinant 1 is called the *special linear group*, denoted $SL_2(R)$. Prove or disprove that $SL_2(R)$ is a group under matrix multiplication.

f. Prove or disprove that $GL_2(R)$ and $SL_2(R)$ are groups under matrix addition.