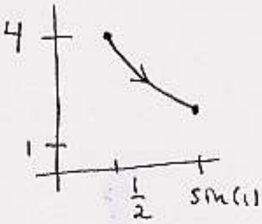
Calculus III: Solutions to Practice Exam I

I (a) $y = \csc^2 t = \frac{1}{\sin^2 t} = \frac{1}{x^2}$ and $\sin(\frac{\pi}{6}) \le x \le \sin(1)$ with x increasing

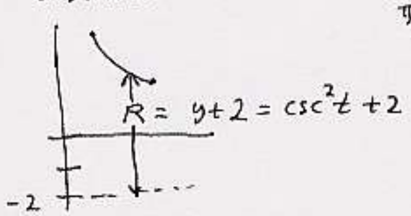


(b) $\frac{dx}{dt} = \cos t$, $\frac{dy}{dt} = 2\csc(t)\cdot(-\csc(t)\cdot\cot(t)) = -2\csc^2 t\cot(t)$

$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{\cos^2 t + 4\csc^4 t\cot^2 t}\,dt$

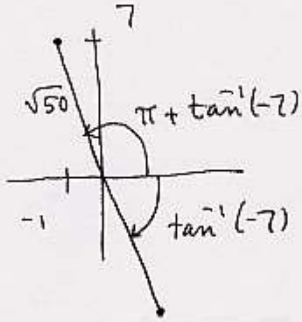
distance traveled $= \int_{\pi/6}^{1} \sqrt{\cos^2 t + 4\csc^4 t\cot^2 t}\,dt$

(c)



$\int_{\frac{\pi}{6}}^{1} 2\pi\,(\csc^2 t + 2)\sqrt{\cos^2 t + 4\csc^4 t\cot^2 t}\,dt$

II



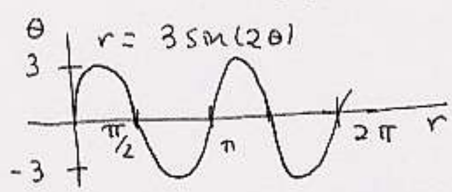
$\left(\sqrt{50},\ \pi + \tan^{-1}(-7) + 2n\pi\right)$

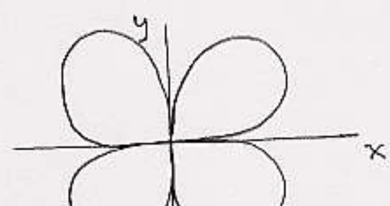
$\left(-\sqrt{50},\ \tan^{-1}(-7) + 2n\pi\right)$,

where n is any integer

III $\frac{dy}{dx} = \frac{\frac{d(r\sin\theta)}{d\theta}}{\frac{d(r\cos\theta)}{d\theta}} = \frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta}$

IV





area $= \int_0^{\pi/2} \frac{1}{2}(3\sin(2\theta))^2\,d\theta = \frac{9}{2}\int_0^{\pi/2} \sin^2(2\theta)\,d\theta = \frac{9}{2}\int_0^{\frac{\pi}{2}} \frac{1}{2} - \frac{1}{2}\cos(4\theta)\,d\theta$

$= \frac{9}{4}\theta - \frac{9}{16}\sin(4\theta)\Big|_0^{\pi/2} = \frac{9\pi}{8}$

V For $\sum_{n=1}^{\infty} (-1)^n n^2$, $s_1 = a_1 = (-1)1^2 = -1$
$s_2 = a_1 + a_2 = (-1)1 + 2^2 = 3$
$s_4 = a_1 + a_2 + a_3 + a_4 = -1 + 4 - 9 + 16 = 10$

VI $\lim_{n\to\infty} \sin\left(\frac{\sqrt{n}}{\sqrt{n}+1}\right) = \lim_{n\to\infty} \sin\left(\frac{1}{1+\frac{1}{\sqrt{n}}}\right) = \sin\left(\frac{1}{1+0}\right) = \sin(1).$

Since this is not 0, the series $\sum_{n=1}^{\infty} \sin\left(\frac{\sqrt{n}}{\sqrt{n}+1}\right)$ diverges

VII $\sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{4}\right)^n = \sum_{n=0}^{\infty} \left(-\frac{3}{4}\right)^n$ is geometric with $r = -\frac{3}{4}$,
so it converges to $\frac{1}{1-(-\frac{3}{4})} = \frac{4}{7}$

VIII $\sum_{n=0}^{\infty} (-1)^n (x-3)^n = \sum_{n=0}^{\infty} (3-x)^n$ geometric, converges
when $|3-x| < 1$
$-1 < 3-x < 1$
$-4 < -x < -2$