Calculus III: Homework Solutions

Answers to odd-numbered textbook exercises can be found in the back of the textbook (and consequently are not included here).

Section 11.7

Solutions to Even-Numbered Textbook Exercises:

10. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(x-1)^n}\right| < 1 \Rightarrow |x-1|\sqrt{\lim_{n\to\infty}\frac{n}{n+1}} < 1 \Rightarrow |x-1| < 1$
$\Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2$; when $x = 0$ we have $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/2}}$, a conditionally convergent series; when $x = 2$ we have $\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$, a divergent series
(a) the radius is 1; the interval of convergence is $0 \le x < 2$
(b) the interval of absolute convergence is $0 < x < 2$
(c) the series converges conditionally at $x = 0$

22. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{\ln(n+1)x^{n+1}}{x^n \ln n}\right| < 1 \Rightarrow |x|\lim_{n\to\infty}\left|\frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)}\right| < 1 \Rightarrow |x|\lim_{n\to\infty}\left(\frac{n}{n+1}\right) < 1 \Rightarrow |x| < 1$
$\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty}(-1)^n \ln n$, a divergent series by the nth-Term Test since $\lim_{n\to\infty}\ln n \ne 0$; when $x = 1$ we have $\sum_{n=1}^{\infty}\ln n$, a divergent series
(a) the radius is 1; the interval of convergence is $-1 < x < 1$
(b) the interval of absolute convergence is $-1 < x < 1$
(c) there are no values for which the series converges conditionally

36. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(\ln x)^{n+1}}{(\ln x)^n}\right| < 1 \Rightarrow |\ln x| < 1 \Rightarrow -1 < \ln x < 1 \Rightarrow e^{-1} < x < e$; when $x = e^{-1}$ or e we obtain the series $\sum_{n=0}^{\infty}1^n$ and $\sum_{n=0}^{\infty}(-1)^n$ which both diverge; the interval of convergence is $e^{-1} < x < e$;
$\sum_{n=0}^{\infty}(\ln x)^n = \frac{1}{1-\ln x}$ when $e^{-1} < x < e$

Section 11.8

Solutions to Even-Numbered Textbook Exercises:

6. $f(x) = \cos x,\ f'(x) = -\sin x,\ f''(x) = -\cos x,\ f'''(x) = \sin x;\ f\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$,
$f'\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}},\ f''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}},\ f'''\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \Rightarrow P_0(x) = \frac{1}{\sqrt{2}}$,
$P_1(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right),\ P_2(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x-\frac{\pi}{4}\right)^2$,
$P_3(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x-\frac{\pi}{4}\right)^2 + \frac{1}{6\sqrt{2}}\left(x-\frac{\pi}{4}\right)^3$

12. $f(x)=(1-x)^{-1} \Rightarrow f'(x)=(1-x)^{-2}, f''(x)=2(1-x)^{-3}, f'''(x)=3!(1-x)^{-4} \Rightarrow \ldots f^{(k)}(x)$
$=k!(1-x)^{-k-1}$; $f(0)=1, f'(0)=1, f''(0)=2, f'''(0)=3!, \ldots, f^{(k)}(0)=k!$
$\Rightarrow \frac{1}{1-x}=1+x+x^2+x^3+\ldots=\sum_{n=0}^{\infty} x^n$

26. $f(x)=\frac{x}{1-x} \Rightarrow f'(x)=(1-x)^{-2}, f''(x)=2(1-x)^{-3}, f'''(x)=3!\,(1-x)^{-4} \Rightarrow f^{(n)}(x)=n!\,(1-x)^{-n-1}$;
$f(0)=0, f'(0)=1, f''(0)=2, f'''(0)=3! \Rightarrow \frac{x}{1-x}=x+x^2+x^3+\ldots=\sum_{n=0}^{\infty} x^{n+1}$

<u>Section 11.9</u>

<u>Solutions to Even-Numbered Textbook Exercises:</u>

2. $e^x=1+x+\frac{x^2}{2!}+\ldots=\sum_{n=0}^{\infty}\frac{x^n}{n!} \Rightarrow e^{-x/2}=1+\left(\frac{-x}{2}\right)+\frac{\left(-\frac{x}{2}\right)^2}{2!}+\ldots=1-\frac{x}{2}+\frac{x^2}{2^2 2!}-\frac{x^3}{2^3 3!}+\ldots$
$=\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^n n!}$

8. $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow x^2\sin x=x^2\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+3}}{(2n+1)!}=x^3-\frac{x^5}{3!}+\frac{x^7}{5!}-\frac{x^9}{7!}+\ldots$

12. $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow x^2\cos(x^2)=x^2\sum_{n=0}^{\infty}\frac{(-1)^n (x^2)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{(2n)!}=x^2-\frac{x^6}{2!}+\frac{x^{10}}{4!}-\frac{x^{14}}{6!}+\ldots$