Modern Algebra I: Group Theory Exam I Review

Disclaimer: as with all review sheets, this is not a comprehensive list of everything you are expected to know (but is a good place to start)!

1. You should be familiar with all of our definitions and theorems (and their proofs) thus far, but, in particular, you should be able to state (and prove) the following from memory:

   Definitions:
   - Binary operation
   - Group
   - Subgroup
   - Order of a group element
   - Order of a group
   - Center of a group
   - Centralizer of a group element
   - Isomorphism

   Theorems:
   - The cancellation law

2. You should have an encyclopedic knowledge of the following examples of groups (that is, you should immediately know its order, elements, operation, identity, the form of its inverses, whether it is abelian, etc):
   - $D_3, S_3, C_3, D_4, V_4, C_4$
   - $(\mathbb{Z}, +), (\mathbb{Q}, +), (n\mathbb{Z}, +), (\mathbb{Z}_n, +_n), (\mathbb{Z}_n^\times, \cdot_n)$
   - $(\mathbb{R}, +), (\mathbb{R}_{>0}, \cdot), (\mathbb{R}^\times, \cdot)$
   - $M_2(\mathbb{R})$ with matrix addition
   - $GL(2, \mathbb{R})$ and $GL(2, \mathbb{Z}_2)$ with matrix multiplication

3. You should be able to prove that something is (or is not) a subgroup, both in the finite and infinite case, by using the respective Subgroup Tests.

4. You should be able to prove or disprove that two given group are isomorphic (see class notes and the practice problems that follow for examples):
   - Finite case (using operation tables)
   - Infinite case and general case (define a mapping and proving that it is bijective and operation-preserving)
   - Showing the two groups are not isomorphic by finding important properties that they do not share

5. You should be able to use the formal definition of isomorphism in proofs (as in theorems 14-17, for example).

Modern Algebra I: Group Theory Practice Problems

1. Prove that the center of a group *G*, *Z(G)*, is a subgroup of G using the Subgroup Test. That is, by showing that *Z(G)* is closed (i.e. if $a, b \in Z(G)$ then $ab \in Z(G)$ and then showing that *Z(G)* contains all of its inverses (i.e. if $a \in Z(G)$ then $a^{-1} \in Z(G)$).

2. Prove that the following groups are isomorphic by (1) defining a bijective function $\phi: G \to H$, (2) proving that ϕ is one-one (injective), (3) proving that ϕ is onto (surjective), and (4) proving that ϕ is operation preserving. (*Note:* for small finite groups in which the mapping is defined by literally assigning each element of G to an element of H, the fact that it is a bijection is obvious; operation-preservation can be shown by comparing similarly aligned operation tables in this case.)
   a. $GL_2(\mathbb{Z}_2) \cong S_3$ (with matrix multiplication and composition, respectively)
   b. $(\mathbb{Z}, +) \cong (n\mathbb{Z}, +)$

3. Are the following groups isomorphic? Prove your assertions.
   a. $(\mathbb{Z}_{10}^{\times}, \cdot_{10})$ and $(\mathbb{Z}_{12}^{\times}, \cdot_{12})$
   b. $(\mathbb{Z}_{10}^{\times}, \cdot_{10})$ and $(\mathbb{Z}_{8}^{\times}, \cdot_{8})$
   c. $(\mathbb{Z}_{8}^{\times}, \cdot_{8})$ and $(\mathbb{Z}_{12}^{\times}, \cdot_{12})$