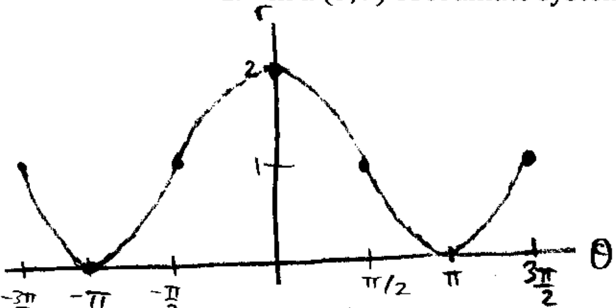
Mathematics 2243: Calculus and Analytic Geometry III
Examination 1
February 1, 2013

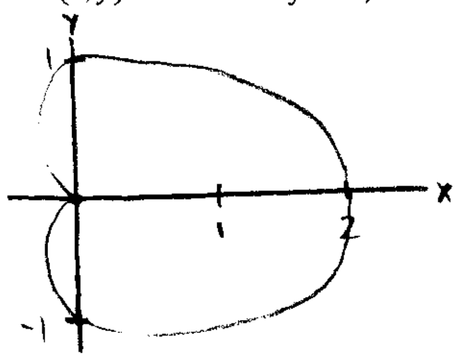
Name: Solutions

I. Consider the polar equation $r = 1 + \cos\theta$. [5 points]

1. In a (θ, r) coordinate system, sketch the graph of the equation $r = 1 + \cos\theta$.



2. In an (x, y) coordinate system, sketch the graph of the polar equation $r = 1 + \cos\theta$.



II. Using $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$, calculate a general formula for $\frac{dy}{dx}$ for a polar curve in which r is a function of θ. [4 points]

$x = r\cos\theta \quad y = r\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

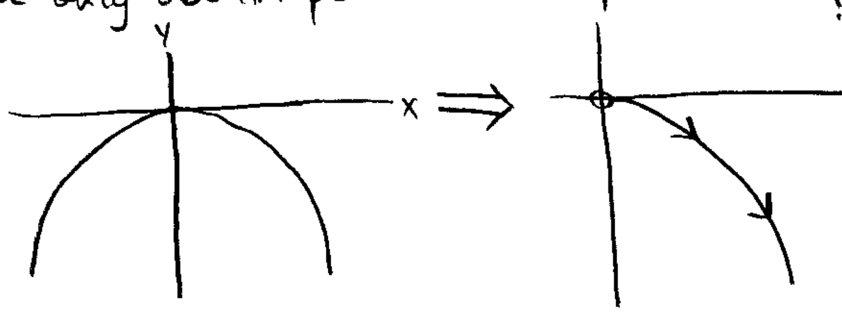
III. For the parametric equations $x = e^t, y =$, solve for t in terms of x and y to eliminate the parameter t (that is, to obtain an xy-equation). Graph the equation in an (x,y) coordinate system, indicating the orientation as t ranges from $-\infty$ to ∞. (Hint: you may use the fact that $\ln(a^b) = b\ln a$.) [6 points]

$x = e^t \Rightarrow \ln x = \ln e^t \Rightarrow t = \ln x, \quad y = -e^{2t} = -(e^t)^2 = -x^2$

$\therefore$ xy equation: $y = -x^2$

Since $x = e^t > 0$, we only obtain points with positive x-coordinates

and x increases as t increases



IV. On the ellipse $x = \cos t,\ y = \sqrt{3}\sin t$, there are two points where the tangent line has slope 1. [6 points]

1. Calculate an expression for $\frac{dy}{dx}$ in terms of t.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sqrt{3}\cos t}{-\sin t}$$

2. Find a value of t between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ where the tangent line has slope 1. i.e. $\frac{dy}{dx} = 1$

$$-\frac{\sqrt{3}\cos t}{\sin t} = 1 \implies \tan t = -\sqrt{3}$$

$$\implies t = -\frac{\pi}{3}$$

3. Find the exact x and y coordinates where that tangent line occurs on the ellipse.

Plug in $t = -\frac{\pi}{3}$: $x = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$

$y = \sqrt{3}\sin\left(-\frac{\pi}{3}\right) = -\frac{3}{2}$

$$(x,y) = \left(\frac{1}{2}, -\frac{3}{2}\right)$$

V. Calculate, if possible, the exact value of the given series. [4 points]

$$\sum_{n=0}^{\infty} \frac{1}{e^{2n}} = 1 + \frac{1}{e^2} + \frac{1}{e^4} + \ldots$$

Geometric series with $r = \frac{1}{e^2}$. Clearly $|r| < 1$. So...

$$\sum_{n=0}^{\infty} \frac{1}{e^{2n}} = \frac{a}{1-r} = \frac{1}{1-\frac{1}{e^2}} = \frac{e^2}{e^2-1}$$

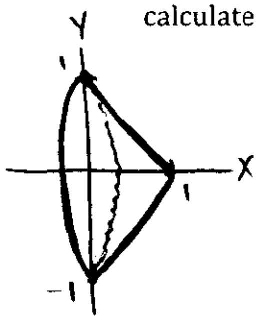
VI. Consider the line segment $x = t, y = 1 - t, 0 \le t \le 1$. It runs from $(0,1)$ to $(1,0)$. [6 points]

1. Use ds and an integral to verify that the length of this segment is $\sqrt{2}$.

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{1^2 + (-1)^2}\,dt = \sqrt{2}\,dt$$

$$\text{length} = \int_0^1 ds = \int_0^1 \sqrt{2}\,dt = \sqrt{2}$$

2. Sketch the cone obtained by rotating this line segment about the x axis. Use ds and an integral to calculate the surface area of this cone.



$$\text{surface area (rotating about x-axis)} = \int_a^b 2\pi y\,ds = \int_0^1 2\pi(1-t)\underbrace{\sqrt{2}\,dt}_{\text{from (1)}}$$

$$= \ldots = \pi\sqrt{2}$$

VII. Consider the line segment $x + y = 1, 0 \le x \le 1$. [5 points]

1. Change the equation $x + y = 1$ to polar coordinates, and solve for r in terms of θ.

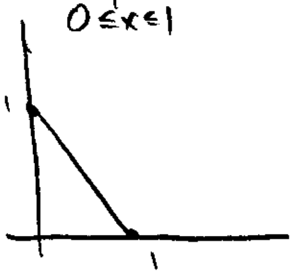
$$r\cos\theta + r\sin\theta = 1$$

$$r(\cos\theta + \sin\theta) = 1$$

$$r = \frac{1}{\cos\theta + \sin\theta}$$

2. Write an integral in terms of θ whose value is the area of the triangle bounded by the line segment and the x and y axes. Supply the limits of integration, but do *not* evaluate the integral.

$x+y=1$, is the same line segment from $\boxed{\text{VI}}$
$0 \le x \le 1$



Bounds on θ: $0 \le \theta \le \frac{\pi}{2}$

$$\text{Area} = \int_\alpha^\beta \frac{1}{2} r^2\,d\theta = \int_0^{\pi/2} \frac{1}{2}\left(\frac{1}{\cos\theta + \sin\theta}\right)^2 d\theta$$

VIII. Give a precise mathematical definition of what it means to say: [up to 2 bonus points]

1. The sequence a_n converges to A.... if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ s.t for all $n \ge N$: $|a_n - A| < \varepsilon$.

2. The series $\sum_{n=0}^{\infty} a_n$ converges to L.... if $\lim_{n\to\infty} S_n = L$

where $S_n = a_1 + a_2 + \ldots + a_n$

IX. Determine if the given sequence converges (and, if so, find its limit). If the sequence diverges, determine if it diverges to $+\infty, -\infty$, or neither. Give some explanation of indication of your reasoning. In particular, when the series is one of the basic types considered in class, such as n^p or r^n, note this and use your knowledge of the behavior of these sequences to determine the limit. [10 points]

1. $a_n = \dfrac{\sqrt{n}}{n+1}$

$\lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{n+1} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}+\frac{1}{\sqrt{n}}} = 0$ (degree numerator < degree denominator)

$\therefore a_n$ converges to 0

2. $b_n = \dfrac{1}{(.999999999999739)^n}$

$\lim\limits_{n\to\infty} b_n = \infty$ since it is r^n with $r>1$

$= \Bigg(\underbrace{\dfrac{1}{.999999999999739}}_{r}\Bigg)^n$

$\therefore b_n$ diverges to ∞

3. $c_n = \dfrac{\sqrt{n}}{1+\sqrt{n}}$

$\lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{1+\sqrt{n}} = 1$

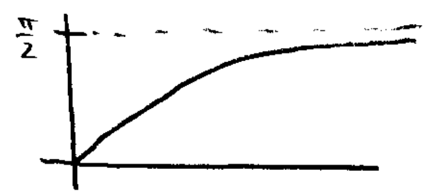
$\therefore c_n$ converges to 1

4. $x_n = n^{-0.000000000001}$

$\lim\limits_{n\to\infty} x_n = 0$ since it is n^p, $p<0$

5. $y_n = \tan^{-1} n$

$\lim\limits_{n\to\infty} \tan^{-1} n = \frac{\pi}{2}$ since $\lim\limits_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$:



X. Explain how one knows immediately that the following series diverges: [4 points + 1 bonus]

$$\sum_{n=1}^{\infty} \frac{n^2}{2n+3n^2}$$

$\lim\limits_{n\to\infty} \dfrac{n^2}{2n+3n^2} = \lim\limits_{n\to\infty} \dfrac{1}{\frac{2}{n}+3} = \frac{1}{3} \neq 0 \quad \therefore$ series diverges

True/false: If $\lim_{n\to\infty} a_n = 0$ for some infinite series $\sum_{n=0}^{\infty} a_n$, does this imply convergence? Explain or give a counterexample.

False!/no! For example, $\sum\limits_{n=0}^{\infty} \frac{1}{n}$ diverges