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Modern Algebra I: Group Theory Midterm Exam

1. Group Fundamentals. Be sure to write precise proofs and justify each of your steps.

    a. Write a precise mathematical definition for group: [5 points]

    b. Let $(G,\cdot)$ be a group and $a \in G$. Use the definition of group to prove that $a \cdot a = a$ if and only if a is the identity of G. [5 points]

    c. State and prove (using the definition of group) the cancellation law. [5 points]

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2. Determine if each of the following statements is true or false. Give a counterexample for each false statement. [13 points]

   a. The center of a group must be abelian.

   b. The centralizer of a group element must be abelian.

   c. If a subgroup of a group is abelian, then the larger group is abelian.

   d. D_3 is isomorphic to $(\mathbb{Z}_{14}^{\times}, \cdot_{14})$

   e. $(\mathbb{Z}, +)$ is isomorphic to $(n\mathbb{Z}, +)$.

   f. If $ab = ca$ in a group, then $b = c$.

   g. There is essentially one group of order three.

   h. There is essentially one group of order four.

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3. Give an example of each of the following (or explain why one does not exist): [7 points + 2 bonus]

   a. An infinite group that is not commutative.

   b. A finite group that is commutative.

   c. Two subgroups (of the same group) that are not isomorphic to each other even though they have the same number of elements.

   d. *Bonus*: a binary operation that is commutative but not associative.

4. For each of the following, prove or disprove that it is a group. You may assume that the operations are associative. [15 points]

   a. The set {-2, 0, 2} with regular addition.

   b. The set {-1, 0, 1} with regular multiplication.

   c. The set {-1, 1} with regular multiplication.

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5. This problem involves the concept of subgroup: [20 points]

   a. Write a precise mathematical definition for subgroup.

   b. Prove that $(n\mathbb{Z}, +), n \in \mathbb{N}$, is a subgroup of $(\mathbb{Z}, +)$.

   c. Let G be a group, and let $a \in G$. Show that the centralizer of a, $C_G(a) = \{g \in G: ag = ga\}$, is a subgroup of G.

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6. Let $(G,\cdot)$ and $(H,*)$ be groups and $\phi: G \to H$ an isomorphism: [20 points]

   a. Write a precise mathematical definition for isomorphism.

   b. Let e_G be the identity of G and e_H the identity of H. Prove that $\phi(e_G) = e_H$.

   c. Let $a, b \in G$. Prove that if a is the inverse of b, then $\phi(a)$ is the inverse of $\phi(b)$.

   d. Let $a \in G$. Prove that if $|a| = 2$ then $|\phi(a)| = 2$.

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7. Prove or disprove that the following groups are isomorphic: [15 points]

   a. V_4 (the symmetries of a non-square rectangle with composition) and $(\mathbb{Z}_{12}^{\times}, \cdot_{12})$.

   b. $(\mathbb{R}_{>0}, \cdot)$ and $(\mathbb{R}, +)$.