Modern Algebra I: Group Theory Homework 8

30. Is $(\mathbb{Q}, +)$ cyclic? Prove your assertion (from the definition, i.e. if it is cyclic, find a generator and prove that it generates the whole group; if not, prove that there can be no such generator).

31. Use the results of the previous problem to prove or disprove that $(\mathbb{Q}, +)$ is isomorphic to $(\mathbb{Z}, +)$.

32. **Definition:** If $(G,\cdot)$ and $(H,*)$ are groups, define a new group $G \times H$ (called the *direct product* or *direct sum*) as follows:
$$G \times H = \{(g, h): g \in G, h \in H\}$$
with the new operation defined *component-wise*:
$$(g_1, h_1)(g_2, h_2) = (g_1 \cdot g_2, h_1 * h_2)$$
(a) Prove that $G \times H$ is a group.
(b) Prove that $G \times H$ is isomorphic to $H \times G$.

33. Prove that C_n (the group of planar rotations of a regular n-gon) is isomorphic to $(\mathbb{Z}_n, +_n)$.

34. **Definition:** Let G be a group and H a nonempty subset of G. For $a \in G$, the set $\{ah: h \in H\}$ is denoted by aH (similarly, when the operation is on the right, $Ha = \{ah: h \in H\}$). When H is a subgroup of G, aH is called the **left coset of *H* in *G* containing *a***. Similarly, Ha is called the **right coset of *H* in *G* containing *a***. For each of the following, find all of the left and right cosets of H in G.
    a. $G = S_3$ and $H = \{(1), (13)\}$
    b. $G = S_3$ and $H = \langle(123)\rangle$ (the subgroup generated by (123))
    c. $G = D_4$ and $H = \{e, R^2\}$
    d. $G = (\mathbb{Z}_9, +_9)$ and $H = \{0, 3, 6\}$
    e. $G = GL(2, \mathbb{Z}_2)$ and $H = SL(2, \mathbb{Z}_2)$

35. Use the results of the previous exercise to answer and make conjectures about the following:
    a. Does $aH = bH$ imply that $a = b$?
    b. Do any of the distinct cosets overlap? That is, is $aH \cap bH$ empty or nonempty for $aH \neq bH$?
    c. Are any of the cosets subgroups themselves? Formulate a conjecture about the conditions needed for a coset to be a subgroup.

36. Let H be a subgroup of G and $a, b \in G$. Prove the following:
    a. $a \in aH$
    b. $aH = H$ if and only if $a \in H$
    c. $(ab)H = a(bH)$ and $H(ab) = (Ha)b$
    d. $aH = bH$ (that is, two cosets are the same) if and only if $a \in bH$
    e. $aH = bH$ if and only if $a^{-1}b \in H$