Modern Algebra I: Group Theory Homework 7

24. In this problem, we will use the Subgroup Test to show that the center of a group G, $Z(G)$, is a subgroup of G.
    a. Show that $Z(G)$ is closed (i.e. if $a, b \in Z(G)$ then $ab \in Z(G)$).
    b. Show that $Z(G)$ contains all of its inverses (i.e. if $a \in Z(G)$ then $a^{-1} \in Z(G)$).

25. Using a method similar to the previous problem, show that the centralizer of a, $C_G(a) = \{g \in G: ag = ga\}$, is a subgroup of G.

26. Determine if the following groups are isomorphic. If so, (1) define an appropriate bijective mapping $\phi: G \to H$, (2) prove that ϕ is one-one (injective), (3) prove that ϕ is onto (surjective), and (4) prove that ϕ is operation preserving. (*Note:* for small finite groups in which the mapping is defined by literally assigning each element of G to an element of H, the fact that it is a bijection is obvious; operation-preservation can be shown by comparing similarly aligned operation tables in this case.) If the groups are not isomorphic, prove your claim.
    a. $(\mathbb{R}_{>0}, \cdot)$ and $(\mathbb{R}, +)$
    b. $GL_2(\mathbb{Z}_2)$ and D_3 (with matrix multiplication and composition, respectively)
    c. $(\mathbb{Z}_{10}^{\times}, \cdot_{10})$ and $(\mathbb{Z}_8^{\times}, \cdot_8)$
    d. $(n\mathbb{Z}, +)$ and $(\mathbb{Z}, +)$
    e. $(\mathbb{Z}_4, +_4)$ and $(\mathbb{Z}_5^{\times}, \cdot_5)$
    f. S_3 and $(\mathbb{Z}_{14}^{\times}, \cdot_{14})$
    a. $(\mathbb{Z}_{10}^{\times}, \cdot_{10})$ and $(\mathbb{Z}_{12}^{\times}, \cdot_{12})$

27. Prove that if $f: A \to B$ and $g: B \to C$ are isomorphisms, then $g \circ f: A \to C$ is an isomorphism (by proving first that that $g \circ f$ is one-one and onto, and then proving that it preserves the operations).

28. **Definition**: A *cyclic group* is a group that is comprised completely of powers of one element, i.e. there exists a $g \in G$ such that $G = \{g^n: n \in \mathbb{Z}\}$. If G is an additive group, this becomes $G = \{ng: n \in \mathbb{Z}\}$. The element g is called the *generator* and the shorthand for "G is generated by g" is $G = \langle g \rangle$.
    a. Determine if the following groups are cyclic (and, if so, state all possible generators): $(\mathbb{Z}_6, +_6)$, $(\mathbb{Z}_6^{\times}, \cdot_6)$, $(\mathbb{Z}_8, +_8)$, $(\mathbb{Z}_8^{\times}, \cdot_8)$, $(\mathbb{Z}_{10}, +_{10})$, $(\mathbb{Z}_{10}^{\times}, \cdot_{10})$
    b. Prove that all cyclic groups are abelian.
    c. Find all of the cyclic subgroups of D_3. Does D_3 have any subgroups that are not cyclic?

29. Suppose G and H are isomorphic groups. Prove that if G is cyclic, then H is cyclic.