<u>Modern Algebra I: Group Theory Homework 4</u>

due Tuesday, January 29th

11. Let a, b, and c be elements of a group such that $ab = ca$. Does it follow that $b = c$? Prove your assertion. (It may be helpful to investigate this matter in a group with a small number of elements, like the symmetries of a triangle.)

12. Prove or disprove that the following are groups. You may assume all operations are associative.
    a. {1, -1, *i*, -*i*} with complex multiplication ($i = \sqrt{-1}$). These are the fourth complex roots of 1 (called the fourth roots of unity). *(Hint: use an operation table.)*
    b. The set of odd integers with addition.
    c. The set of even integers with addition.
    d. Q^+ (the positive rationals) with multiplication.

13. **Definition:** Let *G* be a group and $a \in G$. The *centralizer* of an element *a*, denoted $C_G(a)$, is $\{g \in G : ag = ga\}$. That is, the centralizer of an element is the set of all group elements that commute with it).
    a. Find the *centralizer* of each element of D_4 (the group of symmetries of a square). *Hint: use an operation table!*
    b. Create an operation table *for each of the eight centralizers* (you should have eight operation tables!). Are any of these centralizers (sub)groups? Justify your assertion.

14. **Definition:** Let *G* be a group. The center of *G*, denoted $Z(G)$, is $\{ g \in G : gh=hg$ for every $h \in G \}$. That is, the center of a group is the set of elements that commute with *all* elements of a group (note that this is different from the centralizer; the center applies to the entire group, the centralizer refers to a specific element).
    a. Find $Z(D_4)$, the center of D_4.
    b. Create an operation table for $Z(D_4)$. Is it a (sub)group? Justify your assertion.

15. **Definition:** a *permutation* is a bijective function from a finite set to itself. Recall the previous homework problem in which you listed the permutations of {*a*, *b*, *c*}.

    a. Prove that this set of functions is a group under function composition. Hint: for associativity it is easier to prove that function composition is *always* associative (i.e. prove that if *f*: $S \to T$, *g*: $R \to S$, and *h*: $Q \to R$ are functions, then $f \circ (g \circ h) = (f \circ g) \circ h$). For the other properties, make an operation table and explain how the group axioms are verifiable in the table. As always, justify your reasoning.

    b. This group of functions (permutations) is denoted by S_3 and is called the *symmetric group* on 3 elements. Since its elements are permutations, S_3 is also called a permutation group. As it happens, there is a special shorthand notation for writing permutations.

        i. The function that takes $a \mapsto b$, $b \mapsto c$, and $c \mapsto a$ is denoted by (123) since the first element goes the second, the second goes to the third and the third loops back around to go to the first element.

        ii. The function which takes $a \mapsto b$, $b \mapsto a$, and $c \mapsto c$ is denoted by (12) since it takes the first element to the second and the second to the first.

iii. The identity function is just denoted by (1).

List each of the six functions in S_3 in this notation (there are only three left). Use your function diagrams from the previous homework to help!

c. You can use this notation to calculate compositions of these functions as well. (Recall that function composition is read from right to left!) For example: (12) (123) means that 1 goes 2 in the first permutation, and then since 2 goes to 1 in the second permutation, we know that 1 goes to 1 in the composition. Now 2 goes to 3 in the first permutation and 3 goes to itself in the second permutation so we know that 2 goes to 3 in the composition. This is enough to know the answer is (23). Note that 3 goes to 1 in the first function, and 1 goes to 2 in the second – thus, 3 goes to 2 in the composition (in other words, this answer is correct for all elements).

Calculate all 36 compositions of two permutations using this method. Display the results in an operation table for S_3.

16. (a) Complete the following Cayley (operation) table in such a way that *u* becomes the identity element. In how many ways can this be done while satisfying the definition of a group? (That is, how many groups with two elements are possible?) Justify your answer.

*	*u*	*v*
u		
v		

(b) Complete the following Cayley (operation) table in such a way that *a* becomes the identity element. In how many ways can this be done while satisfying the definition of a group? (That is, how many groups with three elements are possible?) Justify your answer.

*	*a*	*b*	*c*
a			
b			
c			