Mathematics 2243: Calculus and Analytic Geometry III
Examination II
February 22, 2013

Name: Solutions

I. For the series $\sum_{n=1}^{\infty} a_n$: [4 points]

1. Define the nth partial sum s_n .

$$S_n = a_1 + \ldots + a_n = \sum_{i=1}^{n} a_i$$

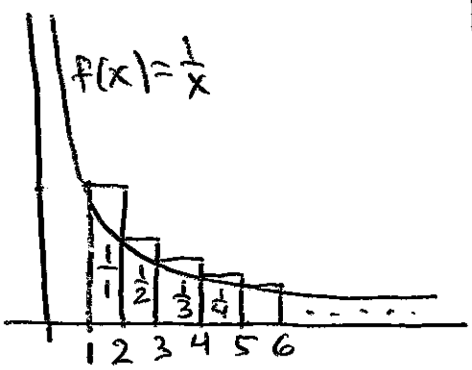
2. Define what it means to say that the series *converges*.

it means $\lim_{n\to\infty} S_n$ exists and is finite

II. Using Riemann sums, prove the divergence of the harmonic series: [4 points]

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Include a sketch in your proof.



Notice that the sketched Riemann sum is given by $\sum_{n=1}^{\infty} \frac{1}{n}$, and that this sum is greater than the integral

$$\int_1^{\infty} \frac{1}{x}\,dx = \left[\ln x\right]_1^{\infty} = \lim_{t\to\infty} \ln t = \infty$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

III. The Alternating Series Test does not apply to any of the following series. For each of the series, tell why it does not apply. [6 points]

1. $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)^2}{n^2}$

$$b_n = \frac{(n+1)^2}{n^2}$$

$$\lim_{n\to\infty} b_n = 1, \text{ not } 0$$

2. $\sum_{n=1}^{\infty} (-1)^n b_n$ where $b_n = \frac{1}{n}$ if n is even, and $b_n = \frac{1}{n^2}$ if n is odd.

$\{b_n\}$ is not a decreasing sequence

IV. Verify that the series [6 points]

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

converges, using the following two methods:

1. By using the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$:

$\lim_{n\to\infty} \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 > 0$, so it behaves the same as $\sum \frac{1}{n^2}$, which converges

2. By using the comparison test with a direct comparison to the terms of $\sum_{n=1}^{\infty} \frac{1}{n^2}$:

$0 < \frac{1}{n^2+1} < \frac{1}{n^2}$ Since $\sum \frac{1}{n^2}$ converges,

so does $\sum \frac{1}{n^2+1}$

V. For each of the following power series, determine the convergence behavior. That is, find the interval of convergence and determine where the convergence is conditional and where it is absolute. Follow any special instructions given. [15 points]

1. $\sum_{n=2}^{\infty} \frac{x^n}{(\ln(n))^n}$ (Use the Root Test.) $\lim_{n\to\infty} \left| \frac{x^n}{[\ln(n)]^n} \right|^{1/n} = \lim_{n\to\infty} \frac{|x|}{\ln n} = 0 < 1$

for all x, so the series converges absolutely for all x

2. $\sum_{n=2}^{\infty} (\ln n)^n x^n$ $\lim_{n\to\infty} \left| \ln(n)^n x^n \right|^{1/n} = |x| \lim_{n\to\infty} \ln(n) = \begin{cases} 0 & \text{if } x=0 \\ \infty & \text{if } x\neq 0 \end{cases}$

So the series converges absolutely for $x=0$ and diverges for all $x\neq 0$

3. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$ (Use the Ratio Test.) $\lim_{n\to\infty} \frac{\left|\frac{(x-1)^{n+1}}{\sqrt{n+1}}\right|}{\left|\frac{(x-1)^n}{\sqrt{n}}\right|} = |x-1| \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = |x-1|$

→ converges absolutely on $0<x<2$, diverges for $x \leq 0, x>2$

→ when $x=2$, the series is a divergent p-series

→ when $x=0$, the series $\sum \frac{(-1)^n}{\sqrt{n}}$ converges by the AST, but the convergence is conditional since $\sum \frac{1}{\sqrt{n}}$ diverges

VI. Suppose that a function $f(x)$ can be written as $c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$ for all x such that $0 < |x| < R$.

[5 points]

1. Show that $c_0 = f(0)$.

$$f(0) = c_0 + c_1(0) + c_2(0)^2 + \ldots = 0$$

2. Use $f'(x)$ to find c_1.

$$f'(x) = c_1 + 2c_2x_2 + 3c_3x^2 + 4c_4x^3 + \ldots$$

$$\boxed{f'(0) = c_1}$$

3. Use $f''(x)$ to find c_2.

$$f''(x) = 2c_2 + 3\cdot 2c_3x + 4\cdot 3x^2 + \ldots$$

$$f''(0) = 2c_2 \Longrightarrow \boxed{c_2 = \frac{f''(0)}{2!}}$$

4. Use $f'''(x)$ to find c_3.

$$f'''(x) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4x + \ldots$$

$$f'''(0) = 3\cdot 2c_3 \Longrightarrow \boxed{c_3 = \frac{f'''(0)}{3!}}$$

5. Use $f^{(4)}(x)$ to find c_4.

$$f^{(4)}(x) = 4\cdot 3\cdot 2c_4 + \ldots$$

$$f^{(4)}(0) = 4\cdot 3\cdot 2c_4 \Longrightarrow \boxed{c_4 = \frac{f^{(4)}(0)}{4!}}$$

6. Use the previous 5 steps to ascertain a formula for c_n.

$$\boxed{c_n = \frac{f^{(n)}(0)}{n!}}$$

VII. For each of the following functions, write the Maclaurin series both as a summation and as an infinite list of terms. For example, $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ [4 points]

1. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

2. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

VIII. Write the general formula for the Taylor series of $f(x)$ at $x = a$. Use the formula to calculate the Taylor series of $\frac{1}{x}$ at a=1. [7 points]

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$f(x) = x^{-1}$ $\quad f(1) = 1$

$f'(x) = -x^{-2}$ $\quad f'(1) = -1$

$f''(x) = 2x^{-3}$ $\quad f''(1) = 2$

$f'''(x) = -3!\,x^{-4}$ $\quad f'''(1) = -3!$

$f^{(4)}(x) = 4!\,x^{-5}$ $\quad f^{(4)}(1) = 4!$

in general, $f^{(n)}(1) = (-1)^n n!$

so... $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!}(x-1)^n = \boxed{\sum_{n=0}^{\infty} (-1)^n (x-1)^n}$

IX. Show that if $0 < b_n < \frac{1}{n}$ for all n, then $\sum_{n=1}^{\infty} \frac{b_n}{n}$ converges. [4 points]

$0 < b_n < \frac{1}{n} \Rightarrow \frac{0}{n} < \frac{b_n}{n} < \frac{1}{n^2} \Rightarrow 0 < \frac{b_n}{n} < \frac{1}{n^2}$

Since $\sum \frac{1}{n^2}$ converges, $\sum \frac{b_n}{n}$ converges