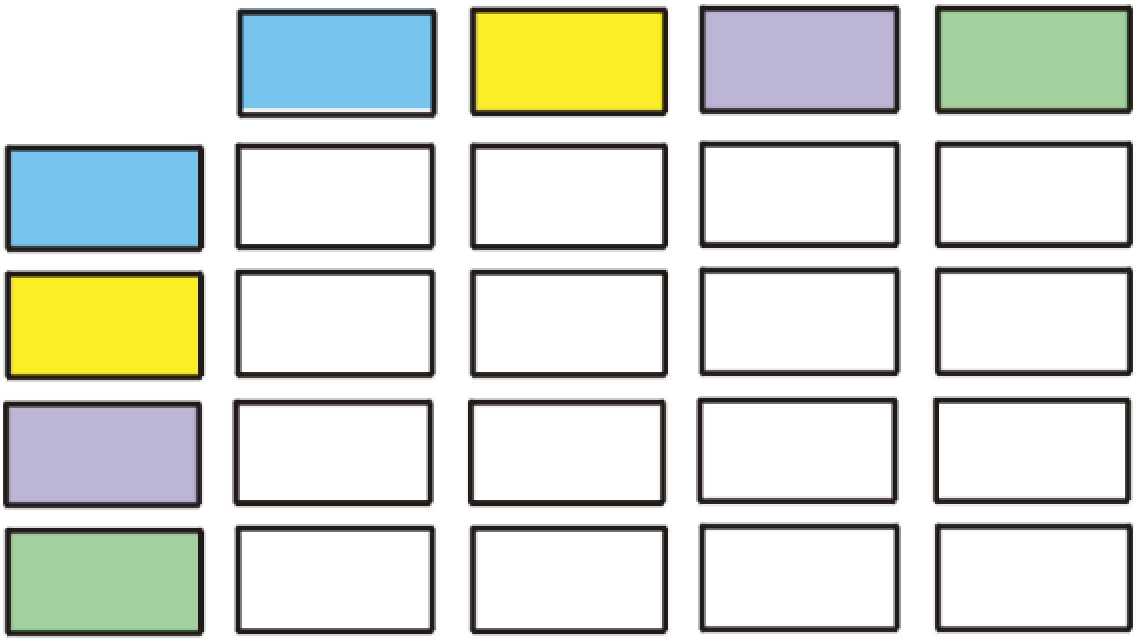
Modern Algebra I: Group Theory

Classwork 3-14

Task 1: Definition: Let G be a group, H a subgroup, and $a \in G$. The set $\{ah: h \in H\}$ is denoted by aH (similarly, when the operation is on the right, $Ha = \{ah: h \in H\}$). When H is a subgroup of G, aH is called the **left coset of *H* in *G* containing *a*.** Similarly, Ha is called the **right coset of *H* in *G* containing *a*.** (The word "coset" is short for "complementary set", indicative of the fact that the cosets are disjoint and their union forms the whole group – these are facts that we will eventually prove.)

For each of the two quotient groups you have formed with D_4 (one of order 2 and the other order 4), write the elements (subsets) in coset notation. That is, write them in the form aH for a given subgroup H. Then fill in the operation tables again, this time using coset notation.

Task 2: Fill out an operation table (using coset notation) for the set of cosets when $H = \{I, FR\}$. Compare this to the quotient group of order 4. Why does $\{I, FR\}$ not form a quotient group while $\{I, R^2\}$ does?



Let's review what we have done with quotient groups and discuss where we are headed:

1. We are trying to figure out what conditions are needed to make a quotient group. Let's summarize what we have found so far:

   (1) When we partition the group we want to use all of the group elements. (It is possible to make a quotient group using only part of the group if the part you break up is a subgroup).
   (2) The subsets that are the elements of our quotient group all have to be the same size. (Try it with different sized subsets at home for fun – enjoy the chaos).
   (3) The subsets that are the elements of our quotient group can't overlap. (Try it with overlapping subsets at home for fun – enjoy the chaos.)
   (4) The identity subset of a quotient group must be a subgroup. (We proved it last time).
   (5) We figured out that the other subsets must be cosets of the subgroup. (We obtained these cosets by multiplying an element of the group by each element of the subgroup).
   (6) We figured out that the cosets of a subgroup will not form a quotient group unless the left and right cosets are the same.

2. A subgroup for which the left and right cosets are the same is called a **normal subgroup.** Here is the formal definition:

   Definition: Let G be a group and H a subgroup. Then H is a **normal subgroup** of G if $gH = Hg$ for all $g \in G$.

3. Notation: For any group G and any subgroup H, G/H is the set of left cosets of H. Formally,

   $$G/H = \{gH : g \text{ is an element of } G\}.$$

   We want to show that normality is a *necessary* and *sufficient* condition for G/H to be a group. In other words, we will prove that G/H is a group and only if H is a normal subgroup of G.