Modern Algebra I: Group Theory Homework 9

38. Recall that the **direct product** (or direct sum) of groups G and H given by $G \times H = \{(g, h): g \in G, h \in H\}$ is also a group. Prove that $G \times H$ is abelian if and only if both G and H are abelian.

39. Create operation tables for the following direct products of groups (with the usual operations). Are these groups isomorphic to any groups with which we have previously worked?
(1) $\mathbb{Z}_2 \times \mathbb{Z}_2$
(2) $\mathbb{Z}_2 \times \mathbb{Z}_3$

40. Prove or disprove that $\mathbb{Z} \times \mathbb{Z}$ is cyclic. (*Hint: suppose that it is cyclic, and examine the consequences.*)

41. Recall that a **normal subgroup** of G is a subgroup H of G for which $aH = Ha$ for all $a \in G$ (that is, a subgroup for which the left cosets are the same as the right cosets). This is denoted as $H \lhd G$. Using this definition, prove that a subgroup H of G is normal in G if and only if $aha^{-1} \in H$ for all $a \in G, h \in H$. (This criterion is often much more useful in practice than the definition for determining if a subgroup is normal).

42. The following exercises involve determining if a subgroup is normal:
(1) Prove that $SL(2, \mathbb{R})$ is a normal subgroup of $GL(2, \mathbb{R})$. (You may need to use the fact that $\det(A^{-1}) = \det(A)^{-1}$.)
(2) Let $H = \{(1), (12)\}$. Is H normal in S_3?
(3) Let $H = \left\{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}: a, b, d \in \mathbb{R}, ad \neq 0\right\}$. Is H normal in $GL(2, \mathbb{R})$?

43. A **group homomorphism** from a group G to a group H is a mapping that preserves the group operation, i.e. $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$ (notice that the only distinction between an isomorphism and a homomorphism is the lack of the bijectivity condition). Define the **kernel** of ϕ to be the set $\ker(\phi) = \{g \in G: \phi(g) = e_h\}$, where e_h is the identity of H. That is, the kernel of a homomorphism is the set of elements that are mapped to the identity. Define the **image** of ϕ to be the set $\operatorname{im}(\phi) = \{h \in H: \phi(g) = h \text{ for some } g \in G\}$. Verify that each of the following are homomorphisms. (Assume the operations are as usual.)
(1) $\phi: GL(2, \mathbb{R}) \to \mathbb{R}^\times$ given by $\phi(A) = \det A$
(2) $\phi: \mathbb{R}^\times \to \mathbb{R}^\times$ given by $\phi(x) = |x|$
(3) $\phi: \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ given by $\phi(x) = 3x$
(4) $\phi: D_4 \to \{1, -1\}$ given by $\phi(x) = 1$ if x is a rotation, -1 if x is a flip (reflection)

44. Find the kernel and image of each of the homomorphisms in the above exercise. Do any of the kernels and images form subgroups?

45. Prove that:
(1) $\ker(\phi)$ is a subgroup of G
(2) $\ker(\phi)$ is a *normal* subgroup of G
(3) $\operatorname{im}(\phi)$ is a subgroup of H. Give an example to show that the image need not be normal.