CHAPTER 5: MULTICOMPLEX SCORE FUNCTION AND INFINITESIMAL PERTURBATION ANALYSIS

The Score Function (SF) method is an effective post-processing method to compute first order probabilistic sensitivities of system response moments or probability-of-failure (POF) with respect to the parameters of the input distributions. Alternatively, the Infinitesimal Perturbation Analysis (IPA) is another technique for calculating first order probabilistic sensitivities. Both the SF and IPA are extended here to compute arbitrary order probabilistic sensitivities, including mixed partial derivatives, through the computation of high order kernel functions and IPA estimators. In regards to the SF method, multicomplex variables are used to numerically compute the high order derivatives of the random variable PDFs with respect to the distribution parameters. For the IPA technique, higher order IPA estimators are computed by evaluating a multicomplex response function. Several numerical examples are used to demonstrate and verify each methodology. All calculations were performed in MATLAB. In the examples presented, the MCX-IPA results indicate that the variances of the probabilistic sensitivities are lower than the variances of the sensitivities computed using MCX-SF method. The results also indicate that the high order probabilistic sensitivities converge with respect to the number of samples at the same rate as standard Monte Carlo estimates.

5.1. Probability of Failure and Statistical Moments of a Response Function

Suppose that a given system response $z(x)$ is a function of $n$ continuous random variables. The random variables are denoted by vector $x$, where $x = \{x_1, x_2, \ldots, x_n\}^T$. If $z = G(x)$ then $G$ is a mapping between random variables $x$ and a new random variable $z$. For example, in linear elastic fracture mechanics, structural failure is assumed to occur when the stress intensity factor exceeds the fracture toughness of the material. In this case the system response, $z$, is the
stress intensity factor. Input random variables may include load and initial crack size. The statistics on these random variables are typically obtained from experimental data.

As with most engineering systems, two important factors in risk-informed decisions are sustainability and reliability. Both need several statistical estimates of the system in order to assess and quantify risk.

In probabilistic analysis, the mean of the system response, $\mu_Z$ is defined as

$$\mu_Z = \int \ldots \int z(x) \cdot f_X(x; \theta) \, dx$$

$$= E[z(x)]$$

(5.1)

where $f_X(x; \theta)$ is the joint probability distribution (JPDF), $\theta$ is a vector of density function parameters, and $E[\cdot]$ is the expected value operator. If the random variables are independent of each other then the JPDF is defined as the product of individual PDF’s (i.e., $f_X = f_{X_1}f_{X_2} \ldots f_{X_n}$).

The variance of the system response is defined as

$$V_Z = \int \ldots \int (z(x) - \mu_Z)^2 \cdot f_X(x; \theta) \, dx$$

$$= E[(z(x) - \mu_Z)^2]$$

(5.2)

Expanding the integrand in Eq. (5.2) and exploiting the linear operator property of the expected value gives

$$V_Z = \int \ldots \int z(x)^2 \cdot f_X(x; \theta) \, dx - \mu_Z^2$$

$$= E[z(x)^2] - \mu_Z^2$$

(5.3)

The corresponding standard deviation of the system response $\sigma_Z$ is

$$\sigma_Z = \sqrt{V_Z}$$

(5.4)
An important metric in reliability analysis is the probability of failure, \( P_f \). The POF is the probability that the response function will exceed or fall short of a specified threshold level. This probability is defined by an \( n \)-dimensional integral over the failure domain. The POF is given as

\[
P_f = \int_{\text{failure}} f_X(x; \theta) \, dx
\]

where failure is defined when \( g(x) \leq 0 \). Generally, in the cases of higher dimensions, the limits of integration pertaining to the failure region are not known. So for higher dimensions the POF is sometimes written in terms of the indicator function \( I(x) \).

\[
P_f = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} I(x) f_X(x; \theta) \, dx = E[I(x)]
\]

The indicator function equals one when \( g(x) \leq 0 \) and zero otherwise.

\[
I(x) = \begin{cases} 
0 & g(x) > 0 \quad \text{safe} \\
1 & g(x) \leq 0 \quad \text{fail}
\end{cases}
\]

### 5.2. Score Function Method

A brief overview of the SF method, as applied to independent random variables, is presented in this section. A more detailed discussion can be found in Millwater (2009). In short, the SF method uses analytical kernel functions, developed from a probability density function (PDF), to calculate sensitivities of the POF and system response moments with respect to parameters of the PDF.

The \( n \)th order kernel function is defined generically as

\[
\kappa^{(n)}_\theta = \frac{\partial^n f_X(x)}{\partial \theta^n} \frac{1}{f_X(x)}
\]
where $\theta$ is any parameter of random variable $X$. Using Eq. (5.8), the $n$th order derivative of the mean response of a system with respect to a parameter of the random variable $i$, denoted as $\theta_i$, can be written as

$$
\frac{\partial^n \mu_Z}{\partial \theta_i^n} = \int_{-\infty}^{\infty} z(x) \cdot k^{(n)}_{\theta_i}(x_i) \cdot f_X(x; \theta) \, dx
$$

$$
= E\left[ z(x_i) \cdot k^{(n)}_{\theta_i}(x_i) \right] \tag{5.9}
$$

$$
= \frac{1}{N} \sum_{k=1}^{N} z(x_k) \cdot k^{(n)}_{\theta_i}(x_{i,k})
$$

where $N$ is the number of samples, $x_{i,k}$ is the $k$th sample of the $i$th random variable, and $z(x_k)$ is the response evaluated at $x_k$.

Similarly, the $n$th order derivative of POF with respect to $\theta_i$ is expressed in terms of the kernel function as

$$
\frac{\partial^n P_f}{\partial \theta_i^n} = \int_{-\infty}^{\infty} I(x) \cdot k^{(n)}_{\theta_i}(x_i) \cdot f_X(x; \theta) \, dx
$$

$$
= E\left[ I(x_i) \cdot k^{(n)}_{\theta_i}(x_i) \right] \tag{5.10}
$$

$$
= \frac{1}{N} \sum_{k=1}^{N} I(x_k) \cdot k^{(n)}_{\theta_i}(x_{i,k})
$$

The first order derivative of the response variance of a system is

$$
\frac{\partial V_Z}{\partial \theta_i} = \int_{-\infty}^{\infty} z(x)^2 \cdot k^{(1)}_{\theta_i}(x_i) \cdot f_X(x; \theta) \, dx - 2\mu_Z \frac{\partial \mu_Z}{\partial \theta_i}
$$

$$
= E[z(x_i)^2 \cdot k^{(1)}_{\theta_i}(x_i)] - 2\mu_Z \cdot E[z(x_i) \cdot k^{(1)}_{\theta_i}(x_i)] \tag{5.11}
$$

$$
= \frac{1}{N} \sum_{i=1}^{N} z(x_{i,k})^2 \cdot k^{(1)}_{\theta_i}(x_{i,k}) - \frac{2\mu_Z}{N} \sum_{i=1}^{N} z(x_{i,k}) \cdot k^{(1)}_{\theta_i}(x_{i,k})
$$

Using the relationship $\sigma_Z^2 = V_Z$, the first order derivative of the response standard deviation is
Higher order partial derivatives of the variance and standard deviation are obtained by repeatedly differentiating Eq’s. (5.11-5.12). The first, second, third, and fourth order derivatives of the mean, variance, and standard deviation of the response are given in Appendix B.

5.3. Multicomplex Score Function Method

The MCX-SF method computes arbitrary order derivatives of a statistical estimate with respect to parameters of a PDF through multicomplex mathematics. Like the SF method, the MCX-SF method can be integrated with any sampling method. Thus, the higher order sensitivities are calculated numerically using suitable modifications of the SF method. The derivatives are calculated from a multicomplex extension of the governing sampling equations that are used to estimate the POF and statistical moments. However, in order to do so, the PDF’s must be holomorphic in the parameter(s) of interest.

For higher order partial derivatives of the statistical estimates with respect to $\theta_i$, a multicomplex perturbation must be added to the parameter $\theta_i$ accordingly. The multicomplex perturbation depends on the order of the derivative to be computed. Let $[\theta_i]$ represent the matrix form of the corresponding perturbed parameter. The multicomplex-valued (matrix form) POF and response moments are shown below.

$$[P_f([\theta_i])] = \frac{1}{N} \sum_{k=1}^{N} \left[ \frac{f_{X_i}(x_{i,k};[\theta_i])}{f_{X_i}} \right]$$  

(5.13)
\[ [\mu_Z(\theta_i)] = \frac{1}{N} \sum_{k=1}^{N} z(x_k) \cdot \frac{[f_{X_i}(x_{i,k};\theta_i)]}{f_{X_i}} \]  
\[ (5.14) \]

\[ [V_Z(\theta_i)] = \frac{1}{N} \sum_{k=1}^{N} z(x_k)^2 \cdot \frac{[f_{X_i}(x_{i,k};\theta_i)]}{f_{X_i}} - [\mu_Z(\theta_i)]^2 \]  
\[ (5.15) \]

where \( f_{X_i} = [f_{X_i}(x_{i,k};\theta_i)]_{d1} \) is the PDF of the \( i \)th random variable and \( [f_{X_i}(x_{i,k};\theta_i)]_{d1} \) is the first element in the matrix-valued PDF. The statistical estimate and its derivatives are located in the first column of the multicomplex statistical estimates (matrix-valued).

The multicomplex standard deviation is calculated as

\[ [\sigma_Z] = SqrtM([V_Z]) \]  
\[ (5.16) \]

where \( SqrtM \) is the matrix square root function.

As an example, a second order derivative of the POF with respect to \( \theta_i \) requires a bicomplex number of the form \( \theta_i + h(i_1 + i_2) \). This bicomplex number is isomorphic to

\[ [\theta_i] = \begin{bmatrix} \theta_i & -h & -h & 0 \\ h & \theta_i & 0 & -h \\ h & 0 & \theta_i & -h \\ 0 & h & h & \theta_i \end{bmatrix} \]  
\[ (5.17) \]

The POF and its derivatives are obtained as follows:

\[ P_f = [P_f(\theta_i)]_{d1} \]  
\[ (5.18) \]

\[ \frac{\partial P_f}{\partial \theta_i} = \frac{[P_f(\theta_i)]_{d1}}{h} = \frac{[P_f(\theta_i)]_{d1}}{h} \]  
\[ (5.19) \]

\[ \frac{\partial^2 P_f}{\partial \theta_i^2} = \frac{[P_f(\theta_i)]_{d1}}{h^2} \]  
\[ (5.20) \]

The second-order mixed partial derivatives are divided into two categories: intra-derivatives and inter-derivatives. Intra-derivatives are those which are differentiated with respect
to the PDF parameters of a random variable (e.g., the mean and standard deviation of a normal distribution, $\frac{\partial^2 P_f}{\partial \mu \partial \sigma}$), while inter-derivatives are differentiated with respect to independent PDF parameters (e.g., the mean of a normal distribution and location of a lognormal distribution, $\frac{\partial^2 P_f}{\partial \mu \partial \lambda}$).

For example, second-order intra-derivatives are determined as follows: Let $\theta_1^i$ and $\theta_1^j$ represent two arbitrary parameters belonging to the same PDF $f_{X_1}(\theta_1^i, \theta_1^j)$. The parameter $\theta_1^i$ is perturbed along the $i_1$ direction and $\theta_1^j$ is perturbed along the $i_2$ direction. Their multicomplex matrix representations are given as

$$[\theta_1^i] = \begin{bmatrix} \theta_1^i & -h & 0 & 0 \\ h & \theta_1^i & 0 & 0 \\ 0 & 0 & \theta_1^j & -h \\ 0 & 0 & h & \theta_1^j \end{bmatrix}$$ (5.21)

$$[\theta_1^j] = \begin{bmatrix} \theta_1^j & 0 & -h & 0 \\ 0 & \theta_1^j & 0 & -h \\ h & 0 & \theta_1^j & 0 \\ 0 & h & 0 & \theta_1^j \end{bmatrix}$$ (5.22)

and the multicomplex POF and multicomplex response moments are written as

$$[P_f([\theta_1^i],[\theta_1^j])] = \frac{1}{N} \sum_{k=1}^{N} I(x_k) \cdot \frac{f_{X_1}(x_{i,k};[\theta_1^i],[\theta_1^j])}{f_{X_1}}$$ (5.23)

$$[\mu_z([\theta_1^i],[\theta_1^j])] = \frac{1}{N} \sum_{k=1}^{N} z(x_k) \cdot \frac{f_{X_1}(x_{i,k};[\theta_1^i],[\theta_1^j])}{f_{X_1}}$$ (5.24)

$$[V_z([\theta_1^i],[\theta_1^j])] = \frac{1}{N} \sum_{k=1}^{N} z(x_k)^2 \cdot \frac{f_{X_1}(x_{i,k};[\theta_1^i],[\theta_1^j])}{f_{X_1}} - [\mu_z([\theta_1^i],[\theta_1^j])]^2$$ (5.25)
where \( f_{X_i} = \left[f_{X_i}(x_{1,i};[\theta_1^i],[\theta_2^i])\right]_{j1} \).

Second-order inter-derivatives are calculated as follows: Let the PDF \( f_{X_i}(\theta_1^i) \) be a function of \( \theta_1^i \) and \( f_{X_2}(\theta_2^i) \) be a function of \( \theta_2^i \). The parameter \( \theta_1^i \) is perturbed along the \( i_1 \) direction (Eq. (5.21)) and \( \theta_2^i \) is perturbed along the \( i_2 \) direction. Substituting \( \theta_1^2 \) in Eq. (5.22) for \( \theta_1^i \), gives the multicomplex matrix representation for \( [\theta_2^i] \). The multicomplex POF and multicomplex response moments are calculated as follows:

\[
[P_f([\theta_1^i],[\theta_2^i])] = \frac{1}{N} \sum_{k=1}^{N} \left( f_{X_i}(x_{1,i};[\theta_1^i]) \right) f_{X_2}(x_{2,i};[\theta_2^i])
\]

\[
[\mu_Z([\theta_1^i],[\theta_2^i])] = \frac{1}{N} \sum_{k=1}^{N} z(x_{k}) \cdot \left( f_{X_i}(x_{1,i};[\theta_1^i]) \right) f_{X_2}(x_{2,i};[\theta_2^i])
\]

\[
[V_Z([\theta_1^i],[\theta_2^i])] = \frac{1}{N} \sum_{k=1}^{N} z(x_{k})^2 \cdot \left( f_{X_i}(x_{1,i};[\theta_1^i]) \right) f_{X_2}(x_{2,i};[\theta_2^i]) - [\mu_Z([\theta_1^i],[\theta_2^i])]^2
\]

where \( f_{X_1} = \left[f_{X_1}(x_{1,i};[\theta_1^i])\right]_{j1} \) and \( f_{X_2} = \left[f_{X_2}(x_{2,i};[\theta_2^i])\right]_{j1} \).

The POF and its derivatives are obtained from the following equations:

\[
P_f = [P_f([\theta_1^i],[\theta_2^i])]_{j1}
\]

\[
\frac{\partial P_f}{\partial \theta_1^i} = \frac{[P_f([\theta_1^i],[\theta_2^i])]_{j2}}{h}
\]

\[
\frac{\partial P_f}{\partial \theta_2^i} = \frac{[P_f([\theta_1^i],[\theta_2^i])]_{j3}}{h}
\]

\[
\frac{\partial^2 P_f}{\partial \theta_1^i \partial \theta_2^i} = \frac{[P_f([\theta_1^i],[\theta_2^i])]_{j4}}{h^2}
\]

The derivatives of the response moments can be obtained similarly by substituting the multiocomplex POF for a multicomplex response moment matrix.
5.4. Implementation for Various Random Variable Distributions

The implementation of multicomplex mathematics to compute derivatives requires functions of matrices. For example, the multicomplex mathematics to compute the kernel function for a second order derivative, \( \frac{\partial^2 f(x, \theta)}{\partial \theta_i^2} \), requires a bicomplex matrix representation for the parameter \( \theta_i \). Therefore, the mathematical evaluation of the PDF must be modified to accept matrix input for the parameters. This process is described below for several probability distributions.

For convenience, the PDFs in this section are formulated in terms of the matrix-valued parameters. A perturbed parameter is equal to its corresponding matrix containing imaginary perturbations. And for an unperturbed parameter, the matrix is equal to the parameter times the identity matrix (e.g., \([\mu] = \mu I\)). Note that since the MCX-SF method uses real samples, the sample \( x_{i,k} \) must also be multiplied times the identity matrix. Non-normal distributions can be defined by either natural parameters or moments. If the moments are perturbed then the appropriate multicomplex natural parameters must be obtained to evaluate the PDF. The relationship between moments and natural parameters for various distribution types are given in the following sections.

A step-by-step outline of the MCX-SF method is given below.

1. Perturb parameter(s) along imaginary unit direction(s).
2. Generate a real-valued random sample, \( x_k \).
3. Evaluate the response function, \( z(x_k) \).
4. Evaluate the multicomplex PDFs, (e.g., \([f_{X_i}(x_{i,k};[\theta_i])]\)).
5. Calculate the multicomplex statistical estimates (matrix form).
6. Extract the POF and the response moments and their derivatives from the first column of the matrix-valued estimates, see Eq’s (5.29-5.32).

5.4.1. Normal Distribution

The PDF for the univariate normal distribution is written as

\[
f(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{(x-\mu)^2}{2\sigma^2} \right]
\]

(5.33)

Its matrix equivalent is

\[
[f(x)] = \frac{I}{[\sigma] \sqrt{2\pi}} \expM \left[ -\frac{(xI-[\mu])^2}{2[\sigma]^2} \right]
\]

(5.34)

where \(\expM\) the exponential of a matrix, \(\expM\), is required.

5.4.2. Bivariate Normal Distribution

The PDF of the bivariate normal distribution is defined as

\[
f(x_1, x_2) = \frac{1}{2\pi \sigma_{x_1} \sigma_{x_2} \sqrt{1-\rho^2}} \exp \left( -\frac{1}{2(1-\rho^2)} \left[ \frac{(x_1-\mu_{x_1})^2}{\sigma_{x_1}^2} + \frac{(x_2-\mu_{x_2})^2}{\sigma_{x_2}^2} - \frac{2\rho (x_1-\mu_{x_1})(x_2-\mu_{x_2})}{\sigma_{x_1} \sigma_{x_2}} \right] \right)
\]

(5.35)

where \(\rho\) is the correlation coefficient. Its matrix extension is

\[
[f(x_1, x_2)] = \frac{I}{2\pi \sigma_{x_1} \sigma_{x_2} |\sqrt{1-\rho^2}|} \cdot \operatorname{ExpM} \left( -\frac{I}{2(1-\rho^2)} \left[ \frac{(x_1I-[\mu_{x_1}])^2}{[\sigma_{x_1}]^2} + \frac{(x_2I-[\mu_{x_2}])^2}{[\sigma_{x_2}]^2} - \frac{2\rho (x_1I-[\mu_{x_1}](x_2I-[\mu_{x_2}])}{[\sigma_{x_1}][\sigma_{x_2}]} \right] \right)
\]

(5.36)

5.4.3. Lognormal Distribution

The PDF of a lognormal distribution is
\[ f(x) = \frac{1}{x\zeta \sqrt{2\pi}} \exp \left[ -\frac{(\ln(x) - \lambda)^2}{2\zeta^2} \right] \]  

(5.37)

where \( \lambda \) and \( \zeta \) are the location and scale parameter (natural parameters) of \( \log x \), respectively.

Its matrix extension is written as

\[ [f(x)] = \frac{I}{x[\zeta] \sqrt{2\pi}} \exp \left[ -\frac{(\ln(x) - \lambda)^2}{2[\zeta]^2} \right] \]  

(5.38)

Derivatives with respect to the mean and standard deviation (moments) can be computed, as follows: Given the matrix form of the mean and standard deviation, the two natural parameters of the lognormal distribution can be computed as

\[ [\lambda] = \log(M([\mu]^2 / SqrtM([\sigma]^2 + [\mu]^2))) \]  

(5.39)

\[ [\zeta] = SqrtM(\log(M([\sigma]^2/[\mu]^2 + I))) \]  

(5.40)

5.4.4. Extreme Value Distribution (Type I)

The PDF of an extreme value distribution (type I) is

\[ f(x) = \frac{1}{b} \exp \left[ -\frac{(x-a)}{b} \right] \cdot \exp \left[ -\exp \left[ -\frac{(x-a)}{b} \right] \right] \]  

(5.41)

where \( a \) and \( b \) are the location and scale parameter (natural parameters), respectively. The matrix extension of the PDF is

\[ [f(x)] = \frac{I}{[b]} \expM \left[ -\frac{(xI-a)}{[b]} \right] \cdot \expM \left[ -\expM \left[ -\frac{(xI-a)}{[b]} \right] \right] \]  

(5.42)

The natural parameters, expressed in terms of the mean and standard deviation (moments), are

\[ [b] = \frac{[\sigma]\sqrt{6}}{\pi} \]  

(5.43)

\[ [a] = [\mu] - [b] \cdot \gamma \]  

(5.44)

where \( \gamma \approx 0.57721 \) is the Euler-Mascheroni constant.
5.4.5. 2-Parameter Weibull Distribution

Another popular distribution is the 2-parameter Weibull distribution. The PDF is defined as

\[ f(x) = \frac{a}{b} \left( \frac{x}{b} \right)^{a-1} \exp \left[ - \left( \frac{x}{b} \right)^a \right] \]  \hspace{1cm} (5.45)

where \( a \) and \( b \) are called the shape and scale parameter (natural parameters), respectively. Its matrix form is given by

\[ f(x) = \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} xI \\ bI \end{bmatrix}^{[a]-I} \exp \left[ - \left( \frac{xI}{bI} \right)^a \right] \]  \hspace{1cm} (5.46)

As shown in the above equation, there are two terms on the right hand side of the equal sign that require the evaluation of a matrix raised to a matrix power. Using the property given in Section 3.5, Equation (5.46) can be rewritten as

\[ f(x) = \begin{bmatrix} a \\ b \end{bmatrix} (H(xI))[M(xI)] \]  \hspace{1cm} (5.47)

where

\[ [H(xI)] = \expM([([a]-I) \cdot \logM(xI/[b])]) \]  \hspace{1cm} (5.48)

\[ [M(xI)] = \expM( \expM([a] \cdot \logM(-xI/[b])) ) \]  \hspace{1cm} (5.49)

The mapping from natural parameters to the moments are given as

\[ [\mu] = [b] \cdot \GammaM(I + I/[a]) \]  \hspace{1cm} (5.50)

\[ [\sigma^2] = [b]^2 \left[ \GammaM(I + 2I/[a]) - (\GammaM(I + I/[a]))^2 \right] \]  \hspace{1cm} (5.51)

where \( \GammaM \) is matrix version of the gamma function, see Section 3.6. The above relationships are not algebraically invertible, so a numerical method to map from moments to natural parameters is needed. For this, the Newton-Raphson iteration method is used.
The Newton-Raphson formula in matrix form is given as

\[
[a]_{n+1} = [a]_n - \frac{[f([a]_n)]}{[f'([a]_n)]}
\]  \hspace{1cm} (5.52)

where \([a]_n\) is the matrix shape parameter at iteration \(n\). The algorithm stops when a metric (e.g., matrix norm), is below a specified tolerance. The function \(f([a]_n)\) is obtained as follows

\[
\frac{[\mu]^2 + [\sigma]^2}{[\mu]^2} = \frac{\Gamma M \left( \frac{2I}{[a]} + I \right)}{\Gamma M \left( \frac{I}{[a]} + I \right)^2}
\]  \hspace{1cm} (5.53)

\[
[f([a]_n)] = \frac{\Gamma M \left( \frac{2I}{[a]} + I \right)}{\Gamma M \left( \frac{I}{[a]} + I \right)^2} - \frac{[\mu]^2 + [\sigma]^2}{[\mu]^2}
\]  \hspace{1cm} (5.54)

The derivative, \(f'([a]_n)\), is given by

\[
[f'([a]_n)] = 2\Gamma M \left( \frac{2I}{[a]} + I \right) \frac{\psi M \left( \frac{I}{[a]} + I \right) - \psi M \left( \frac{2I}{[a]} + I \right)}{[a]_n^2 \Gamma M \left( \frac{I}{[a]} + I \right)^2}
\]  \hspace{1cm} (5.55)

where \(\psi M\) is the matrix version of the digamma function. After determining the matrix shape parameter, Eq. (5.50) is used to determine the multicomplex scale parameter \([b]\).

5.5. Infinitesimal Perturbation Analysis

In the IPA technique, the response moments are mapped to a standard uniform space by using the inverse cumulative density function (CDF). The inverse CDF is a transformation from standard uniform variables \(u\) to random variables \(x\) (i.e., \(x(u;\theta) = F^{-1}(u;\theta)\)), where \(F\) is the CDF.
The mean of the response, in the standard uniform space, is defined as (M. C. Fu, 2013)

\[ \mu_Z = \int_0^1 z(x(u; \theta)) \, du \]
\[ = E[z(x(u; \theta))] \tag{5.56} \]

The \( n \)th order derivative of the response mean with respect to \( \theta_i \), assuming that the order of the differentiation and integration can be interchanged, is written as

\[ \frac{\partial^n \mu_Z}{\partial \theta_i^n} = \int_0^1 \frac{\partial^n z(x(u; \theta))}{\partial \theta_i^n} \, du \]
\[ = E\left[ \frac{\partial^n z(x(u; \theta))}{\partial \theta_i^n} \right] = E\left[ \omega_{\theta_i}^{(n)}(u_i) \right] \tag{5.57} \]
\[ = \frac{1}{N} \sum_{i=1}^N \omega_{\theta_i}^{(n)}(u_{i,k}) \]

where \( \omega_{\theta_i}^{(n)} \) is the \( n \)th order IPA estimator and \( u_{i,k} \) is the \( k \)th realization of the \( i \)th random variable.

The first order IPA estimator is defined generically as (Glasserman, 1991)

\[ \omega_{\theta}^{(1)}(u) = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} \tag{5.58} \]

where \( \partial z / \partial x \) is the derivative of the response function with respect to random variable \( x \) and \( \partial x / \partial \theta \) is the derivative of the inverse CDF with respect to the parameter \( \theta \).

\[ \frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta} F^{-1}(u; \theta) \tag{5.59} \]

The second order IPA estimator is

\[ \omega_{\theta}^{(2)} = \frac{\partial}{\partial \theta} \left[ \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} \right] \tag{5.60} \]
\[ = \frac{\partial^2 z}{\partial x^2} \left( \frac{\partial x}{\partial \theta} \right)^2 + \frac{\partial z}{\partial x} \frac{\partial^2 x}{\partial \theta^2} \]
Similarly, the response variance is written as

$$V_z = \int_{0}^{1} z(x(u; \theta))^2 \, du - \mu_z^2 \hspace{1cm} (5.61)$$

The first order derivative of the response variance is

$$\frac{\partial V_z}{\partial \theta_i} = \int_{0}^{1} 2 \cdot z(x(u; \theta)) \cdot \omega^{(i)}_z(u_i) \, du - 2\mu_z \frac{\partial \mu_z}{\partial \theta_i}$$

$$= E[2 \cdot z(x(u; \theta)) \cdot \omega^{(i)}_z(u_i)] - 2\mu_z \frac{\partial \mu_z}{\partial \theta_i} \hspace{1cm} (5.62)$$

Using the relationship $\sigma_z^2 = V_z$, the corresponding derivative of the standard deviation is

$$\frac{\partial \sigma_z}{\partial \theta_i} = \frac{1}{2\sigma_z} \frac{\partial V_z}{\partial \theta_i} \hspace{1cm} (5.63)$$

To calculate higher order derivatives of the variance and standard deviation analytically, the chain rule must be appropriately applied to Eq. (5.62 and 5.63). This leads to long derivative expressions that are sometimes impractical to calculate. Therefore, a sampling-based MCX-IPA technique, for the computation of higher order derivatives of POF and statistical moments of a response function with respect to the statistical parameters, is presented in the next section.

The following example is used to demonstrate the IPA technique analytically. Consider the case where the response is equal to a single, normally distributed, random variable (i.e., $z = g(x) = x$). The mean and standard deviation parameters are denoted by $\mu$ and $\sigma$, respectively. Therefore, the response mean $\mu_z = \mu$ and the standard deviation $\sigma_z = \sigma$. The analytical derivatives are $\frac{\partial \mu_z}{\partial \mu} = 1$, $\frac{\partial \mu_z}{\partial \sigma} = 0$, $\frac{\partial \sigma_z}{\partial \mu} = 1$, and $\frac{\partial \sigma_z}{\partial \sigma} = 0$.

The inverse CDF of a normally distributed random variable is given as

$$x = \sigma \sqrt{2 \cdot \text{erf}^{-1}(2u - 1)} + \mu \hspace{1cm} (5.64)$$

where $\text{erf}^{-1}$ is the inverse error function. From which,
\[
\frac{\partial x}{\partial \mu} = 1 
\]

\[
\frac{\partial x}{\partial \sigma} = \sqrt{2}\text{erf}^{-1}(2u-1) 
\]

The derivative of the response function is \(\frac{\partial z}{\partial x} = 1\); thus, the first order derivatives of the response mean are

\[
\frac{\partial \mu_z}{\partial \mu} = \int_0^1 \frac{\partial z}{\partial x} \frac{\partial x}{\partial \mu} du \\
= \int_0^1 1 \cdot du \\
= 1 
\]

\[
\frac{\partial \mu_z}{\partial \sigma} = \int_0^1 \frac{\partial z}{\partial x} \frac{\partial x}{\partial \sigma} du \\
= \int_0^1 \sqrt{2}\text{erf}^{-1}(2u-1) \cdot du \\
= 0 
\]

The derivatives of the response variance are calculated as follows:

\[
\frac{\partial V_z}{\partial \mu} = \int_0^1 (2 \cdot x) \cdot \frac{\partial z}{\partial x} \frac{\partial x}{\partial \mu} du - 2\mu_z \frac{\partial \mu_z}{\partial \mu} \\
= \int_0^1 (2 \cdot (\sqrt{2}\text{erf}^{-1}(2u-1) + \mu)) \cdot 1 \cdot 1 \cdot du - 2\mu_z \frac{\partial \mu_z}{\partial \mu} \\
= 2\mu - 2\mu \\
= 0 
\]

\[
\frac{\partial V_z}{\partial \sigma} = \int_0^1 (2 \cdot x) \cdot \sqrt{2}\text{erf}^{-1}(2u-1) \cdot du - 2\mu_z \frac{\partial \mu_z}{\partial \sigma} \\
= \int_0^1 2 \cdot (\sqrt{2}\text{erf}^{-1}(2u-1) + \mu) \cdot 1 \cdot \sqrt{2}\text{erf}^{-1}(2u-1) \cdot du - 2\mu_z \frac{\partial \mu_z}{\partial \sigma} \\
= 2\sigma \int_0^1 (\sqrt{2}\text{erf}^{-1}(2u-1))^2 \cdot du + 2\mu \int_0^1 \sqrt{2}\text{erf}^{-1}(2u-1) \cdot du + 0 \\
= 2\sigma(1) + 0 + 0 
\]

The corresponding response standard deviation sensitivities are
\[
\frac{\partial \sigma_Z}{\partial \mu} = \frac{1}{2\sigma_z} \frac{\partial V_Z}{\partial \mu} = 0 \\
\frac{\partial \sigma_Z}{\partial \sigma} = \frac{1}{2\sigma_z} \frac{\partial V_Z}{\partial \sigma} = 1
\] (5.71) (5.72)

5.6. Multicomplex Infinitesimal Perturbation Analysis

The MCX-IPA technique is an alternative approach to computing response moment derivatives of any order. There is a caveat associated with the MCX-IPA method; the method does not use real samples. So, unlike the MCX-SF method, existing samples that have been stored cannot be reused. That’s because the MCX-IPA method requires multicomplex samples in order to apply the chain rule numerically. This methodology requires the re-coding of any existing sample generating algorithm to handle multicomplex parameter arguments via matrix representation. As a result, multicomplex samples can be generated and used to evaluate the response moments. The output of the response moments is now given as a matrix that contains the values of the response moments and their derivative information.

For instance, higher order derivative information of the mean response is obtained from

\[
[\mu_Z] = \frac{1}{N} \sum_{i=1}^{N} [z_k] 
\] (5.73)

where \([z_k]\) is a multicomplex (matrix form) realization of the response function. These realizations are calculated from

\[
[z_k] = z([x_i, ([\theta_i)])] 
\] (5.74)
In which, \([x_{i,k}(\theta_i)])\) is a multicomplex sample. Notice that the sample is a function of a parameter that has been perturbed along one or more imaginary unit directions, \([\theta_i])\). Consequently, the response function is also matrix-valued.

Multicomplex samples are computed from the inverse CDF of the \(i\)th random variable. If the response function contains other random variables whose parameters were unperturbed, any classical random number generator can be used to obtain “real” samples of those random variables.

The response variance and its derivative information is obtained by

\[
[V_Z] = \frac{1}{N} \sum_{k=1}^{N} [z_k]^2 - [\mu_z]^2
\]  \hspace{1cm} (5.75)

The standard deviation is given as

\[
[\sigma_Z] = \text{SqrtM}(V_Z)
\]  \hspace{1cm} (5.76)

The two main requirements for the MCX-IPA technique are as follows: the response function must be continuously differentiable with respect to the random variables and the function used to generate samples must be holomorphic in the parameters of interest.

### 5.7. Generating Multicomplex Samples

This section presents a method for generating multicomplex samples for the following distribution types: Multivariate normal, lognormal, extreme value (Type I), and 2-parameter Weibull. Although, there are various methods for generating random samples from a specific distribution, the MCX method requires closed-form formulas that are composed of holomorphic functions. In particular, they must be holomorphic in the distribution parameters. As shown in the next sections, each sampling method is expressed in terms of the natural parameters, not their moments (i.e., \(\mu\) and \(\sigma\)).
The lognormal, extreme value (Type I), and 2-parameter Weibull distributions have a mapping (transformation) between their natural parameters and moment parameters. Therefore, the sensitivities of the response moments can either be calculated with respect to the moment parameters or natural parameters. The latter is obtained by perturbing the natural parameters along the imaginary unit directions. The former is obtained by perturbing the moment parameters along the imaginary unit directions. Like the MCX-SF method, the transformation equations must be used to obtain the appropriate natural parameters when the moments of non-normal are perturbed.

A step-by-step outline of the MCX-IPA technique is given below.

1. Perturb parameter(s) along imaginary unit direction(s).
2. Generate “real” samples (unperturbed) and multicomplex sample(s), \([x_{i,k}([θ_i])]\).
3. Evaluate the response function, \([z_k]\).
4. Calculate the multicomplex statistical estimates (matrix form).
5. Extract the response moments and their derivatives from the first column of the matrix-valued estimates; see Eq’s (5.73, 5.74, and 5.76).

The next section demonstrates how to generate multicomplex samples for several distribution types. As in the previous section, a perturbed parameter is equal to its corresponding matrix containing imaginary perturbations. And for an unperturbed parameter, the matrix is equal to the parameter times the identity matrix (e.g., \(μ = μI\)).

5.7.1. Multivariate Normal Distribution

For the multivariate normal distribution random samples are traditionally generated using the following formula:
\[ x = uL^T + \mu \] (5.77)

where \( x \) is a column vector of random variables, \( u \) is a column vector of standard normal random variables, \( L^T \) is the upper triangular matrix resulting from a Cholesky factorization of the covariance matrix \( \Sigma \), and \( \mu \) is a column vector of mean parameters.

The Cholesky factorization is a decomposition of the form \( \Sigma = LL^T \). There are various methods for calculating the decomposition; however, the MCX method requires a decomposition method that can explicitly provide the upper triangular matrix in terms of the statistical parameters. For this, the Cholesky-Banachiewicz (Gentle, 2007) factorization is used. The Cholesky-Banachiewicz formula is:

\[
L_{jj} = \sqrt{\Sigma_{jj} - \sum_{k=1}^{j-1} L_{jk}^2}
\]

\[
L_{ij} = \frac{1}{L_{jj}} \left( \Sigma_{ij} - \sum_{k=1}^{i-1} L_{ik}L_{jk} \right), \quad \text{for } i > j
\] (5.78)

For instance, consider a bivariate normal case where the covariance matrix is

\[
\Sigma = \begin{bmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{12} & \Sigma_{22}
\end{bmatrix}
\] (5.79)

The upper triangular matrix is expressed as

\[
L^T = \begin{bmatrix}
L_{11} & L_{12} \\
0 & L_{22}
\end{bmatrix}
\] (5.80)

where

\[
L_{11} = \sqrt{\Sigma_{11}}
\]

\[
L_{12} = \Sigma_{12} / L_{11}
\]

\[
L_{22} = \sqrt{\Sigma_{22} - L_{12}^2}
\] (5.81)
To generate matrix random samples, multicomplex matrices are passed through Eq’s (5.77-5.81). The multicomplex-correlated normal random samples \( [x] \) are computed using the following formula:

\[
[x] = [u][L^T] + [\mu]
\] (5.82)

where \( [x] = [x_1, x_2, \ldots, x_n] \) denotes column matrix of random variables, \( [u] = [u_1 I, u_2 I, \ldots, u_n I] \) denotes a column matrix of standard normal random variables times the identity matrix, and \( [L^T] \) matrix extension of the upper triangular matrix. For a bivariate case \( [L^T] \) is given as:

\[
[L^T] = \begin{bmatrix}
L_{11} & L_{12} \\
0 & L_{22}
\end{bmatrix}
\] (5.83)

where

\[
L_{11} = \text{SqrtM}([\Sigma_{11}]) \\
L_{12} = [\Sigma_{12}] / [L_{11}] \\
L_{22} = \text{SqrtM}([\Sigma_{22}] - [L_{12}]^2)
\] (5.84)

The univariate normal distribution is a special case of the of the multivariate normal distribution. The normal distribution is only defined by the mean \( \mu \) and standard deviation \( \sigma \).

Thus, Eq. (5.82) becomes

\[
[x] = u[\sigma] + [\mu]
\] (5.85)

Alternatively, the matrix samples can be calculated from

\[
[x] = [\sigma] \sqrt{2} \text{erf}^{-1}(2u - 1) + [\mu]
\] (5.86)

where \( u \) is a standard uniform variable and \( \text{erf}^{-1} \) is the inverse error function.

### 5.7.2. Lognormal Distribution

Lognormal samples can be calculated from
\[ [x] = \exp M [m \xi + [\lambda]] \]  
(5.87)

where \([\lambda]\) denotes the matrix location parameter, \([\xi]\) is the matrix scale parameter, and \(\exp M\) is the matrix exponential. The two parameters of the lognormal distribution can be computed using the relationships found in Section 5.4.3.

### 5.7.3. Extreme Value Distribution (Type I)

Extreme Value samples can be calculated using the inverse sampling method as follows:

\[ [x] = [a] - [b] \ln(\ln(u)) \]  
(5.88)

where \([a]\) denotes the matrix location parameter, \([b]\) is the matrix scale parameter, \(u\) is a standard uniform random variable, and \(\ln\) is the natural logarithm. The two parameters of the extreme value distribution can be computed using the relationships found in Section 5.4.4.

### 5.7.4. 2-Parameter Weibull Distribution

Weibull samples can be calculated from

\[ [x] = [b](-\ln(1-u))^{1/[a]} \]  
(5.89)

where \([a]\) denotes the matrix location parameter, \([b]\) is the matrix scale parameter, and \(u\) is a standard uniform random variable. As seen in the above equation, the real value \((-\ln(1-u))\) is raised to a matrix power, \(1/[a]\). In order to evaluate this quantity accurately the matrix logarithm and matrix exponential are utilized. Taking the matrix logarithm of both sides of Eq. (5.94) yields,

\[ \log M([x]) = \log M([b]) + \frac{I}{[a]} \ln(-\ln(1-u)) \]  
(5.90)

Next, taking the matrix exponential of both sides of Eq. (5.95) yields,

\[ \exp M[\log M([x])] = \exp M[\log M([b]) + \frac{I}{[a]} \ln(-\ln(1-u))] \]  
(5.91)
The sampling equation then becomes

\[ [x] = [b] \cdot \exp M \left[ \frac{I}{[a]} \ln(-\ln(1-u)) \right] \]  

(5.92)

Section 5.4.5 details the mapping from natural parameters to the mean and standard.

**Bicomplex Weibull Samples**

In order to better understand the concept of multicomplex samples, one million bicomplex Weibull samples were generated using Eq. (5.97) and analyzed. For this study, the shape parameter was perturbed along the \( i_1 \) direction (Im₁) and the scale parameter was perturbed along the \( i_2 \) direction (Im₂); namely,

\[
[a] = \begin{bmatrix}
a & -h & 0 & 0 \\
h & a & 0 & 0 \\
0 & 0 & a & -h \\
0 & 0 & h & a
\end{bmatrix}
\]  

(5.93)

\[
[b] = \begin{bmatrix}
b & 0 & -h & 0 \\
0 & b & 0 & -h \\
h & 0 & b & 0 \\
0 & h & 0 & b
\end{bmatrix}
\]  

(5.94)

where \( a = 2 \), \( b = 3 \), and \( h = 10^{-10} \).

From an analytical perspective, a bicomplex Weibull sample is related to the natural parameters as follows:

\[
x(u) + h \frac{\partial x(u)}{\partial a} i_1 + h \frac{\partial x(u)}{\partial b} i_2 + h^2 \frac{\partial^2 x(u)}{\partial a \partial b} i_1 i_2 \leftrightarrow [x] = \begin{bmatrix}
x(u) & -h \frac{\partial x(u)}{\partial a} & -h \frac{\partial x(u)}{\partial b} & h^2 \frac{\partial^2 x(u)}{\partial a \partial b} \\
h \frac{\partial x(u)}{\partial a} & x(u) & -h^2 \frac{\partial^2 x(u)}{\partial a \partial b} & -h \frac{\partial x(u)}{\partial b} \\
h \frac{\partial x(u)}{\partial b} & -h^2 \frac{\partial^2 x(u)}{\partial a \partial b} & x(u) & -h \frac{\partial x(u)}{\partial a} \\
h^2 \frac{\partial^2 x(u)}{\partial a \partial b} & h \frac{\partial x(u)}{\partial b} & h \frac{\partial x(u)}{\partial a} & x(u)
\end{bmatrix}
\]  

(5.95)
where \( x \) is the inverse CDF and \( 0 \leq u < 1 \). Hence, it is also possible to generate bicomplex Weibull samples directly from the inverse CDF (real part) and analytical derivatives of its inverse CDF (imaginary parts) with respect to the natural parameters. The analytical inverse CDF and derivative expressions for this example are given as

\[
x(u) = b(-\ln(1-u))^{\frac{1}{a}} \tag{5.96}
\]

\[
\frac{\partial x(u)}{\partial a} = -\frac{b \log(-\log(1-u))(-\ln(1-u))^{\frac{1}{a}}}{a^2} \tag{5.97}
\]

\[
\frac{\partial x(u)}{\partial b} = (-\ln(1-u))^{\frac{1}{a}} \tag{5.98}
\]

\[
\frac{\partial^2 x(u)}{\partial a \partial b} = -\frac{b \log(-\log(1-u))(-\ln(1-u))^{\frac{1}{a}}}{a^2} \tag{5.99}
\]

Figure 5.1 presents the one million samples that were generated using Eq. (5.92) and the analytical solutions. In the figure, the real part of the multicomplex samples are shown on the abscissa and the imaginary parts are shown on the ordinate. Using Figure 5.1, a bicomplex Weibull sample (for the given parameters) is roughly equal to \( 8 - 4hi_1 + 2.5hi_2 - 1.2h^2i_1i_2 \) (see Eq. (5.95)). The imaginary parts of each sample, which are associated with the directions \( i_1 (\left[ x \right]_{I1}) \), \( i_2 (\left[ x \right]_{I2}) \), and \( i_1i_2 (\left[ x \right]_{I1I2}) \), are denoted with blue, magenta, and red markers, respectively. The analytical solutions are denoted with solid black lines.
5.8. Numerical Examples

In this section, several numerical examples are presented to demonstrate a sampling-based probabilistic sensitivity analysis using multicomplex score function estimators (MCX-SF) and multicomplex Infinitesimal Perturbation Analysis (MCX-IPA) estimators. In each example higher order derivatives of the statistical estimates with respect to parameters of input random variables are accurately computed using both MCX-SF method and MCX-IPA sampling-based techniques.

5.8.1. Kernel Functions via MCX Method

Let $x$ denote a random variable that is normally distributed with mean $\mu$ and standard deviation $\sigma$ (i.e., $x \sim N(\mu, \sigma)$). The analytical normal kernels for the mean parameter, up to fourth order, are shown below.
\[ k^{(1)}_{\mu}(x) = \frac{1}{\sigma} U \] (5.100)

\[ k^{(2)}_{\mu}(x) = \frac{1}{\sigma^2} (U^2 - 1) \] (5.101)

\[ k^{(3)}_{\mu}(x) = \frac{1}{\sigma^3} U(U^2 - 3) \] (5.102)

\[ k^{(4)}_{\mu}(x) = \frac{1}{\sigma^4} (U^4 - 6U^2 + 3) \] (5.103)

where \( U = (x - \mu) / \sigma \). The analytical normal kernels for the standard deviation parameter are given below.

\[ k^{(1)}_{\sigma}(x) = \frac{1}{\sigma} (U^2 - 1) \] (5.104)

\[ k^{(2)}_{\sigma}(x) = \frac{1}{\sigma^2} (U^4 - 5U^2 + 2) \] (5.105)

\[ k^{(3)}_{\sigma}(x) = \frac{1}{\sigma^3} (U^6 - 12U^4 + 27U^2 - 6) \] (5.106)

\[ k^{(4)}_{\sigma}(x) = \frac{1}{\sigma^4} (U^8 - 22U^6 + 123U^4 - 168U^2 + 24) \] (5.107)

The following example considers the standard normal distribution. The MCX-SF kernels were calculated using quadcomplex numbers. Since there are two parameters, two separate PDF evaluations were required. As expected, the MCX-SF kernels agree with the analytical kernel functions.

**Calculation of kernel with respect to the mean**

In the first PDF evaluation, the mean was perturbed along the pure imaginary unit directions as such \( \mu + h(i_1 + i_2 + i_3 + i_4) \) and the standard deviation was unperturbed (i.e.,
Let $[\sigma] = \sigma I$. Let $[\mu]$ represent the corresponding 16 by 16 quadcomplex matrix and $I$ the 16 by 16 identity matrix.

$$
[\mu] =
\begin{bmatrix}
\mu & -h & -h & 0 & -h & 0 & 0 & -h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\mu & 0 & -h & 0 & -h & 0 & 0 & -h & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\mu & -h & 0 & 0 & -h & 0 & 0 & -h & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & h & h & \mu & 0 & 0 & 0 & -h & 0 & 0 & 0 & -h & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & -h & -h & 0 & 0 & 0 & 0 & -h & 0 & 0 & 0 & 0 \\
0 & 0 & h & 0 & 0 & h & h & \mu & 0 & -h & 0 & -h & 0 & 0 & 0 & 0 \\
h & 0 & 0 & h & 0 & 0 & 0 & h & h & \mu & -h & -h & 0 & 0 & 0 & -h \\
h & 0 & 0 & 0 & 0 & h & h & \mu & 0 & -h & 0 & -h & 0 & 0 & 0 & 0 \\
0 & 0 & h & 0 & 0 & 0 & 0 & h & h & \mu & -h & 0 & 0 & -h & 0 & 0 \\
0 & 0 & 0 & h & 0 & 0 & h & 0 & 0 & 0 & \mu & -h & -h & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & h & 0 & 0 & h & 0 & 0 & h & \mu & -h & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & h & 0 & 0 & h & 0 & 0 & h & h & \mu & -h & 0 & 0 \\
0 & 0 & 0 & 0 & h & 0 & 0 & h & 0 & 0 & h & h & h & \mu & -h & 0 & 0 
\end{bmatrix}
$$

(5.108)

Using Eq. (5.36), the PDF is

$$
[f(x)] = \frac{I}{(\sigma I)^{\frac{3}{2}}} \exp \left\{ -(xI[\mu])^2 / 2(\sigma I)^2 \right\}
$$

(5.109)

The above multicomplex PDF $[f(x)]$ is a matrix containing the PDF and derivative information. The PDF is approximately equal to the diagonal elements of the matrix (e.g., $[f(x)]_{1,1} = f(x)$). The other elements in the matrix contain derivative information of the PDF with respect to the mean. As in the standard SF method, the normalization factor $[f(x)]_{1,1}$ is introduced.

$$
[K_{\mu}(x)] = \frac{[f(x)]}{[f(x)]_{1,1}}
$$

(5.110)

Note that the diagonal elements of $[K_{\mu}(x)]$ are approximately equal to one. The first, second, third, and fourth order numerical kernels of the mean parameter are obtained as follows:
\[ \kappa_{\mu}^{(1)}(x) = \frac{[K_{\mu}(x)]_{2,1}}{h^2} \]  
(5.111)

\[ \kappa_{\mu}^{(2)}(x) = \frac{[K_{\mu}(x)]_{4,1}}{h^2} \]  
(5.112)

\[ \kappa_{\mu}^{(3)}(x) = \frac{[K_{\mu}(x)]_{8,1}}{h^3} \]  
(5.113)

\[ \kappa_{\mu}^{(4)}(x) = \frac{[K_{\mu}(x)]_{16,1}}{h^4} \]  
(5.114)

The analytical and MCX-SF kernels of the mean parameter times the PDF are shown in Figure 5.1. As demonstrated in the figure, the numerical MCX-SF kernels match with the analytical kernel functions.

Figure 5.2: Kernel Functions With Respect to the Mean of a Standard Normal Distribution

Calculation of the kernel function with respect to the standard deviation
In the second PDF evaluation, the standard deviation is perturbed along the imaginary unit directions \((\sigma + h(i_1 + i_2 + i_3 + i_4))\) and the mean is unperturbed.

\[
[\sigma] = \begin{bmatrix}
\sigma & -h & -h & 0 & -h & 0 & 0 & -h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 h & \sigma & 0 & -h & 0 & -h & 0 & 0 & -h & 0 & 0 & 0 & 0 & 0 & 0 \\
 h & 0 & \sigma & -h & 0 & 0 & -h & 0 & 0 & -h & 0 & 0 & 0 & 0 & 0 \\
 0 & h & h & \sigma & 0 & 0 & 0 & -h & 0 & 0 & -h & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & \sigma & -h & -h & 0 & 0 & 0 & -h & 0 & 0 & 0 & 0 \\
 h & 0 & 0 & 0 & h & \sigma & 0 & -h & 0 & 0 & 0 & -h & 0 & 0 & 0 \\
 0 & h & 0 & 0 & h & 0 & \sigma & 0 & 0 & 0 & 0 & 0 & -h & 0 & 0 \\
 h & 0 & 0 & h & 0 & 0 & 0 & \sigma & -h & -h & 0 & -h & 0 & 0 & -h \\
 0 & h & 0 & 0 & 0 & 0 & 0 & h & \sigma & 0 & -h & 0 & -h & 0 & 0 \\
 0 & 0 & h & 0 & 0 & 0 & 0 & h & 0 & \sigma & -h & 0 & 0 & -h & 0 \\
 0 & 0 & 0 & h & 0 & 0 & 0 & h & 0 & 0 & \sigma & -h & 0 & 0 & -h \\
 0 & 0 & 0 & 0 & 0 & h & 0 & 0 & h & 0 & 0 & \sigma & -h & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & h & 0 & 0 & h & 0 & 0 & h & \sigma
\end{bmatrix} \tag{5.115}
\]

The PDF is written as

\[
[f(x)] = \frac{I}{[\sigma]\sqrt{2\pi}} \exp\left\{-(xI-\mu I)^2 / 2[\sigma]^2\right\} \tag{5.116}
\]

After dividing by the normalization factor, the kernels of the standard deviation can be obtained using Eq’s (5.109-5.112). The analytical and MCX-SF kernels of the standard deviation parameter times the PDF are given in Figure 5.2.
Consider a quadratic response function of the form

\[ z = g(x) = A_0 + A_1 x_1 + A_2 x_2 + B_1 x_1^2 + B_2 x_2^2 + C_{12} x_1 x_2 \]  

(5.117)

where \( x_1 \sim N[\mu_1, \sigma_1 = 1] \) and \( x_2 \sim \text{EVD}[\mu_2 = 1, \sigma_2 = 1] \). The coefficients are \( A_0 = -1, \ A_1 = 2.5, \ A_2 = 1, \ B_1 = 0, \ B_2 = 1, \) and \( C_{12} = -1 \). The first order derivatives of the response moments with respect to parameters \( \mu_1, \sigma_1, \mu_2, \) and \( \sigma_2 \) were computed using a single quad-complex analysis, \( \mathbb{C}_4 \). The parameters \( \mu_1, \sigma_1, \mu_2, \) and \( \sigma_2 \) were perturbed by a small step size, \( h = 10^{-20} \), along the imaginary axis \( i_1, i_2, i_3, \) and \( i_4 \), respectively. As a byproduct, mixed derivatives were also computed from the analysis.
In the first step of the MCX-IPA analysis, appropriate 16 by 16 matrices were created for each parameter. Let \([\mu_1]\) and \([\sigma_1]\) denote the matrices corresponding to \(\mu_1 + h_i\) and \(\sigma_1 + h_i\), respectively. The matrices \([\mu_2]\) and \([\sigma_2]\) correspond to \(\mu_2 + h_i\) and \(\sigma_2 + h_i\), respectively. The matrices are explicitly given below.

\[
[\mu_1] = \begin{bmatrix}
\mu_i & -h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h & \mu_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu_i & -h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & h & \mu_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_i & -h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & h & \mu_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mu_i & -h & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & h & \mu_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_i & -h & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h & \mu_i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_i & -h & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h & \mu_i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_i & -h & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h & \mu_i
\end{bmatrix}
\]

(5.118)
\[
[\sigma_i] = \\
\begin{bmatrix}
\sigma_i & 0 & -h & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sigma_i & 0 & -h & 0 & 0 & 0 & 0 & 0 & 0 \\
h & 0 & \sigma_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & h & 0 & \sigma_i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sigma_i & 0 & -h & 0 & 0 & 0 \\
0 & 0 & 0 & h & 0 & \sigma_i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sigma_i & 0 & -h & 0 \\
0 & 0 & 0 & 0 & 0 & h & 0 & \sigma_i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & h & 0 & \sigma_i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_i 
\end{bmatrix}
\]
(5.119)

\[
[\mu_2] = \\
\begin{bmatrix}
\mu_2 & 0 & 0 & 0 & -h & 0 & 0 & 0 & 0 & 0 \\
0 & \mu_2 & 0 & 0 & 0 & -h & 0 & 0 & 0 & 0 \\
0 & 0 & \mu_2 & 0 & 0 & 0 & -h & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_2 & 0 & 0 & 0 & -h & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & 0 & -h & 0 \\
0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & 0 & -h \\
0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & -h \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & -h \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_2
\end{bmatrix}
\]
(5.120)
This example includes a non-normally distributed random variable. As a result in order to obtain the sensitivities of interest, \((\mu_1, \sigma_1, \mu_2, \text{and } \sigma_2)\), all non-normal distributions require an additional step to convert from moments to natural parameters. The appropriate multicomplex-valued natural parameters for the extreme value distribution (Section 5.4.4) were computed from the following formulas:

\[
[b] = \frac{[\sigma_2] \sqrt{6}}{\pi} \quad (5.122)
\]

\[
[a] = [\mu_2] - [b] \cdot \gamma \quad (5.123)
\]

Note that if the natural parameters were perturbed (sensitivities with respect to the natural parameters) no further steps are required. Although the MCX-SF and MCX-IPA methodologies are different, the pre-processing procedures are the same. Both techniques utilize the same multicomplex-valued natural parameters within their methodology.

**MCX-SF**
The multicomplex normal PDF (Section 5.4.1) is given as

\[
f(x_1) = \frac{1}{\sigma_1 \sqrt{2\pi}} \exp\left[ -\frac{(x_1 - [\mu_1])^2}{2[\sigma_1]^2} \right]
\]  

(5.124)

and the multicomplex extreme value PDF (Section 5.4.4) is

\[
f(x_2) = \frac{1}{b} \exp\left[ -\frac{(x_2 - [a])}{b} \right] \cdot \exp\left[ -\exp\left[ -\frac{(x_2 - [a])}{b} \right] \right]
\]  

(5.125)

Substituting Eq. (5.122) and Eq. (5.123) into the sampling equations yield,

\[
[\mu_Z] = \frac{1}{N} \sum_{k=1}^{N} \left[ \frac{f(x_1)}{f(x_2)} \cdot \frac{f(x_1)}{f(x_2)} \right] [f(x_1)] [f(x_2)]
\]  

(5.126)

\[
[V_Z] = \frac{1}{N} \sum_{k=1}^{N} \left[ \frac{f(x_1)}{f(x_2)} \cdot \frac{f(x_1)}{f(x_2)} \right] \left[ [\mu_Z]^2 \right]
\]  

(5.127)

where \( f(x_1) = [f(x_1)]_1 \) and \( f(x_2) = [f(x_2)]_1 \). The corresponding matrix-valued standard deviation is calculated from \( [\sigma_Z] = \text{SqrtM}([V_Z]) \).

**MCX-IPA**

The normal samples were computed using

\[
[x_{1,k}] = [\sigma_1] m_k + [\mu_1]
\]  

(5.128)

where \( m_k \) is a standard normal sample and \( [x_{1,k}] \) is a resulting quad-complex normal sample.

Alternatively, these samples could have been generated using the following normal inverse CDF:

\[
[x_{1,k}] = [\sigma_1] \sqrt{2} \text{erf}^{-1}(2u_k - 1) + [\mu_1]
\]  

(5.129)

where \( u_k \) is a standard uniform sample.

The extreme value random samples were computed using the inverse sampling method as follows:

\[
[x_{2,k}] = [a] - [b] \cdot \ln(-\ln(u_k))
\]  

(5.130)
where \([x_{2,k}]\) is resulting quadcomplex extreme value sample.

Substituting both Eq. (5.127) and Eq. (128) into Eq. (5.115) yields,

\[
[z_k] = A_0 I + A_1 [x_{1,k}] + A_2 [x_{2,k}] + B_1 [x_{1,k}]^2 + B_2 [x_{2,k}]^2 + C_{12} [x_{1,k}] [x_{2,k}] \quad (5.131)
\]

where \([z_k]\) is the quad-complex realization of the response function and \(I\) denotes a 16 by 16 identity matrix. The multicomplex response moments were estimated using Eq. (5.78), Eq. (5.80), and Eq. (5.81).

**Results**

Let \(T\) denote a generic probabilistic measure (e.g., \(\mu_z\)) and \([T]\) denote a corresponding generic multicomplex probabilistic measure (e.g., \([\mu_z]\)). The probabilistic measure and the corresponding sensitivity estimates were extracted as follows:

Response moment:

\[
T = [T]_{1,1} \quad (5.132)
\]

First order derivatives:

\[
\frac{\partial T}{\partial \mu_1} = \frac{[T]_{2,1}}{h}, \quad \frac{\partial T}{\partial \sigma_1} = \frac{[T]_{3,1}}{h}, \quad \frac{\partial T}{\partial \mu_2} = \frac{[T]_{5,1}}{h}, \quad \frac{\partial T}{\partial \sigma_2} = \frac{[T]_{9,1}}{h} \quad (5.133)
\]

Second order derivatives:

\[
\frac{\partial^2 T}{\partial \mu_1 \partial \sigma_1} = \frac{[T]_{4,1}}{h^2}, \quad \frac{\partial^2 T}{\partial \mu_1 \partial \mu_2} = \frac{[T]_{6,1}}{h^2}, \quad \frac{\partial^2 T}{\partial \mu_2 \partial \sigma_1} = \frac{[T]_{7,1}}{h^2} \quad (5.134)
\]

\[
\frac{\partial^2 T}{\partial \mu_1 \partial \sigma_2} = \frac{[T]_{10,1}}{h^2}, \quad \frac{\partial^2 T}{\partial \sigma_1 \partial \sigma_2} = \frac{[T]_{11,1}}{h^2}, \quad \frac{\partial^2 T}{\partial \mu_2 \partial \sigma_2} = \frac{[T]_{13,1}}{h^2}
\]

Third order derivatives:

\[
\frac{\partial^3 T}{\partial \mu_1 \partial \mu_2 \partial \sigma_1} = \frac{[T]_{8,1}}{h^3}, \quad \frac{\partial^3 T}{\partial \mu_1 \partial \sigma_1 \partial \sigma_2} = \frac{[T]_{12,1}}{h^3} \quad (5.135)
\]
\[
\frac{\partial^3 T}{\partial \mu_1 \partial \mu_2 \partial \sigma_2} \approx \frac{[T]_{14,1}}{h^3}; \quad \frac{\partial^3 T}{\partial \mu_2 \partial \sigma_1 \partial \sigma_2} \approx \frac{[T]_{15,1}}{h^3}
\]

Fourth order derivative:

\[
\frac{\partial^4 T}{\partial \mu_1 \partial \mu_2 \partial \sigma_1 \partial \sigma_4} \approx \frac{[T]_{16,1}}{h^4}
\]

(5.136)

The MCX-SF, MCX-IPA, and semi-analytic sensitivities are given in Table 5.1 and 5.2. The semi-analytic solutions (reference derivatives) were calculated from the integrals governing the statistical moments of the response. In MATLAB, the derivatives of the integrands were numerically integrated using the function \textit{integral2}.

Table 5.1 contains the derivatives of the response mean with respect to the input parameters. As shown, most of the higher order derivatives in Table 5.2 are zero; however, derivatives of the standard deviation of the response (Table 5.3) are not. So Table 5.2 does not provide much insight into the inherent characteristics of the MCX-SF method and MCX-IPA technique. Conversely, a conclusion can be drawn from Table 5.3: at \(10^7\) samples the accuracy of the MCX-IPA is much better than the accuracy of the MCX-SF when compared to the semi-analytic solutions.

Table 5.1: Comparison of the Higher Order Sensitivities of the Response Mean for the Quadratic Response Function

<table>
<thead>
<tr>
<th>Derivative</th>
<th>MCX-SF</th>
<th>MCX-IPA</th>
<th>Semi-Analytic</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{\partial \mu_2}{\partial \mu_1})</td>
<td>1.50262</td>
<td>1.50019</td>
<td>1.50000</td>
</tr>
<tr>
<td>(\frac{\partial \mu_2}{\partial \sigma_1})</td>
<td>-0.00189</td>
<td>-0.00086</td>
<td>0.00000</td>
</tr>
<tr>
<td>(\frac{\partial \mu_2}{\partial \mu_2})</td>
<td>1.99956</td>
<td>1.99992</td>
<td>2.00000</td>
</tr>
<tr>
<td>(\frac{\partial \mu_2}{\partial \sigma_2})</td>
<td>2.00286</td>
<td>1.99738</td>
<td>2.00000</td>
</tr>
</tbody>
</table>

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<table>
<thead>
<tr>
<th>Derivative</th>
<th>MCX-SF</th>
<th>MCX-IPA</th>
<th>Semi-Analytic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\partial \sigma_{Z} / \partial \mu_{1}$</td>
<td>-0.773576</td>
<td>-0.77935</td>
<td>-0.77923</td>
</tr>
<tr>
<td>$\partial \sigma_{Z} / \partial \sigma_{1}$</td>
<td>0.79975</td>
<td>0.80743</td>
<td>0.80743</td>
</tr>
<tr>
<td>$\partial \sigma_{Z} / \partial \mu_{2}$</td>
<td>1.18775</td>
<td>1.18598</td>
<td>1.18745</td>
</tr>
<tr>
<td>$\partial \sigma_{Z} / \partial \sigma_{2}$</td>
<td>5.12031</td>
<td>5.13141</td>
<td>5.12646</td>
</tr>
<tr>
<td>$\partial^{2} \sigma_{Z} / \partial \mu_{1} \partial \sigma_{1}$</td>
<td>0.16759</td>
<td>0.15597</td>
<td>0.15654</td>
</tr>
<tr>
<td>$\partial^{2} \sigma_{Z} / \partial \mu_{1} \partial \mu_{2}$</td>
<td>-0.26260</td>
<td>-0.26647</td>
<td>-0.26745</td>
</tr>
<tr>
<td>$\partial^{2} \sigma_{Z} / \partial \mu_{2} \partial \sigma_{1}$</td>
<td>-0.99078</td>
<td>-0.98249</td>
<td>-0.98326</td>
</tr>
</tbody>
</table>

*10^7 samples

Table 5.2: Comparison of the Higher Order Derivatives of the Response Standard Deviation for the Quadratic Response Function
As implied earlier, for a specified number of samples the MCX-IPA derivatives are more accurate than the MCX-SF derivatives. To better illustrate this behavior, the variance of the fourth order derivative as a function of the number of samples is given in Figure 5.1. The variance of the derivative, for a fixed number of samples, was calculated using one thousand samples of the derivative estimate.
Fourth Order Derivative: \[ \frac{\partial^4 \sigma_Z}{\partial \mu_1 \partial \mu_2 \partial \sigma_1 \partial \sigma_2} \]

**Figure 5.4: Quadratic Response: Variance of the Derivatives**

The MCX-SF and MCX-IPA derivatives are denoted with a red dashed line and a solid blue line, respectively. As seen in Figure 5.3, due to the inherent IPA characteristics, the variance of the derivative computed from the MCX-IPA is much lower than the variance of the derivative computed via the MCX-SF method for a specified number of samples. For example, to achieve a variance of roughly $10^{-2}$, the MCX-IPA requires roughly $10^3$ samples, while the MCX-SF method requires about $10^7$ samples. The MCX-IPA derivatives converge much faster than the MCX-SF derivatives. It is also observed that variance of the fourth order derivative varies as a function of $1/N$ in both cases.

Another characteristic different between the two sensitivity approaches is computational cost. This next study investigates the computational time needed to compute a fourth order sensitivity, $\partial^4 \sigma_Z / \partial \mu_1 \partial \mu_2 \partial \sigma_1 \partial \sigma_2$, using both the MCX-SF and MCX-IPA technique. The results
are only an indication of how this problem behaves, since the problem statement dictates the type of multicomplex functions to be evaluated.

In order to alleviate computationally expensive Monte Carlo simulations, MATLAB’s parallel “for loops” command, \textit{parfor}, was utilized. A \textit{parfor} loop allows the user to execute an iterative loop across multiple workers simultaneously, which are called \textit{labs}. A total of 8 \textit{labs} were used in this study. There is overhead in calling \textit{parfor}, so MATLAB’s stopwatch timer \textit{tic} was started once the \textit{parfor} loop began and the elapsed time was displayed using \textit{toc} once the code completed. Figure 5.4 presents the time it took to calculate $\frac{\partial^4 \sigma_z}{\partial \mu_1 \partial \mu_2 \partial \sigma_1 \partial \sigma_2}$ using both approaches. The majority of the pre-processing and post-processing in both codes are the same. The difference between both scripts is how the samples are generated and the multicomplex functions evaluated.

![Figure 5.5: CPU Time Versus Number of Samples](image-url)
5.8.3. Linear Response Function with Correlated Normal Random Variables

This problem seeks to demonstrate and verify the MCX-SF method and MCX-IPA technique for problems containing correlated normal random variables. Consider a linear response function of the form

\[ z = g(x) = x_1 - x_2 \]  \hspace{1cm} (5.137)

where \( x_1 \) and \( x_2 \) are correlated Gaussian (normal) distributed random variables. The statistics are \( \mu = [10,3], \sigma = [5,2] \), and \( \rho = 0.9 \). The covariance matrix of a bivariate normal distribution is

\[ \Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \]  \hspace{1cm} (5.138)

where \( \rho \) is the correlation coefficient between \( x_1 \) and \( x_2 \). In this example, first; second; third; and fourth order derivatives of statistical estimates with respect to the input parameters are calculated using various MCX methods.

It is well known that a linear response function of \( n \) Gaussian random variables provides an exact solution to the POF and statistical moments of the response (Ang & Tang, 2007), thus exact sensitivities can be computed for comparison. The analytical solution of POF is given as

\[ P_f = \Phi(-\beta) \]  \hspace{1cm} (5.139)

where \( \Phi \) is the standard normal CDF and the reliability index \( \beta \) is

\[ \beta = \frac{\mu_z}{\sigma_z} \]  \hspace{1cm} (5.140)

For this problem the closed form solutions of the statistical moments are

\[ \mu_z = \mu_1 - \mu_2 \]  \hspace{1cm} (5.141)

and

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\[ \sigma_Z = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho \sigma_1 \sigma_2} \]  

(5.142)

The exact sensitivities of the response moments with respect to a desired input parameter can be determined by differentiating Eq’s. (5.139-5.140). Likewise, the derivatives of the POF can also be determined. The analytical first, second, third, and fourth order derivatives of \( P_f \) with respect to the parameter \( \theta_j \) are given as follows:

\[ \frac{\partial P_f}{\partial \theta_j} = \frac{\partial \Phi(B)}{\partial \theta_j} = \phi B_{\theta_j} \]  

(5.143)

\[ \frac{\partial^2 P_f}{\partial \theta_j^2} = \frac{\partial}{\partial \theta_j} \left[ \frac{\partial P_f}{\partial \theta_j} \right] = \phi B_{\theta_j} \left( B_{\theta_j} \right)^2 + \phi B_{\theta_{\rho\theta_j}} \]  

(5.144)

\[ \frac{\partial^3 P_f}{\partial \theta_j^3} = \frac{\partial}{\partial \theta_j} \left[ \frac{\partial^2 P_f}{\partial \theta_j^2} \right] = \phi B_{\theta_j} \left( B_{\theta_j} \right)^3 + 3 \phi B_{\theta_j} B_{\theta_{\rho\theta_j}} + \phi B_{\theta_{\rho\theta_{\rho\theta_j}}} \]  

(5.145)

\[ \frac{\partial^4 P_f}{\partial \theta_j^4} = \frac{\partial}{\partial \theta_j} \left[ \frac{\partial^3 P_f}{\partial \theta_j^3} \right] = \phi B_{\theta_j} \left( B_{\theta_j} \right)^4 + 6 \phi B_{\theta_j} \left( B_{\theta_j} \right)^2 + 3 \phi B_{\theta_j} \left( B_{\theta_{\rho\theta_j}} \right)^2 + 4 \phi B_{\theta_j} B_{\theta_{\rho\theta_j}} B_{\theta_{\rho\rho\theta_j}} + \phi B_{\theta_{\rho\rho\rho\rho\theta_j}} \]  

(5.146)

The variable \( B = -\beta \) and \( \phi \) is the standard normal PDF. The derivatives of both functions are expressed using subscripts. In which, \( B_{\theta_j} = \partial B / \partial \theta_j \); \( B_{\theta_{\rho\theta_j}} = \partial^2 B / \partial \theta_j^2 \);

\( B_{\theta_{\rho\rho\theta_j}} = \partial^3 B / \partial \theta_j^3 \); \( B_{\theta_{\rho\rho\rho\rho\theta_j}} = \partial^4 B / \partial \theta_j^4 \); \( \phi_{\theta_j} = \partial \phi / \partial \theta_j \); \( \phi_{\theta_{\rho\theta_j}} = \partial^2 \phi / \partial \theta_j^2 \); and \( \phi_{\theta_{\rho\rho\theta_j}} = \partial^3 \phi / \partial \theta_j^3 \).

In this example there are five input parameters, as a result, both the MCX-SF and MCX-IPA require five separate multicomplex function evaluations. In practice each input parameter is separately perturbed along four imaginary unit directions (i.e., quadcomplex analysis) prior to
evaluating the appropriate multicomplex-statistical estimates. In all five cases the multicomplex matrices were similar, meaning that the multicomplex matrix of perturbations for each case were of the form

$$
[\theta_j] = \begin{bmatrix}
\theta_j & -h & -h & 0 & -h & 0 & 0 & -h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h & \theta_j & 0 & -h & 0 & -h & 0 & 0 & -h & 0 & 0 & 0 & 0 & 0 & 0 \\
h & 0 & \theta_j & 0 & 0 & -h & 0 & 0 & -h & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & h & h & \theta_j & 0 & 0 & 0 & -h & 0 & 0 & -h & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \theta_j & -h & -h & 0 & 0 & 0 & 0 & -h & 0 & 0 & 0 \\
h & 0 & 0 & h & \theta_j & 0 & -h & 0 & 0 & 0 & 0 & -h & 0 & 0 & 0 \\
0 & h & 0 & 0 & h & 0 & \theta_j & -h & 0 & 0 & 0 & 0 & -h & 0 & 0 \\
0 & 0 & h & 0 & 0 & h & h & \theta_j & 0 & 0 & 0 & 0 & -h & 0 & 0 \\
0 & 0 & h & 0 & 0 & 0 & 0 & h & 0 & \theta_j & -h & 0 & 0 & 0 & -h \\
0 & 0 & 0 & 0 & h & 0 & 0 & h & 0 & 0 & 0 & h & \theta_j & 0 & 0 \\
0 & 0 & 0 & 0 & h & 0 & 0 & 0 & h & 0 & 0 & h & 0 & \theta_j & -h \\
0 & 0 & 0 & 0 & h & 0 & 0 & h & 0 & 0 & h & h & \theta_j & 0 \\
0 & 0 & 0 & 0 & h & 0 & 0 & h & 0 & 0 & h & h & \theta_j & 0 \\
0 & 0 & 0 & 0 & h & 0 & 0 & h & 0 & 0 & h & h & \theta_j & 0 \\
\end{bmatrix}
$$

(5.147)

All other parameters were equal to their scalar value or scalar value times the identity matrix in order to maintain the appropriate matrix dimension.

**MCX-SF**

The PDF, in its general multicomplex form, is given in Section 5.4.2. The multicomplex statistical moments that were evaluated are

$$
[\mu_z] = \frac{1}{N} \sum_{k=1}^{N} z_k \cdot \frac{[f(x_1, x_2)]}{f(x_1, x_2)} 
$$

(5.148)

$$
[V_z] = \frac{1}{N} \sum_{k=1}^{N} z_k^2 \cdot \frac{[f(x_1, x_2)]}{f(x_1, x_2)} - [\mu_z]^2
$$

(5.149)
where \( f(x_1, x_2) \approx [f(x_1, x_2)]_{i_1} \). The multicomplex standard deviation was determined as such \( SqrtM([V_Z]) \). The multicomplex POF is

\[
[P_j] = \frac{1}{N} \sum_{k=1}^{N} I \cdot \frac{[f(x_1, x_2)]}{f(x_1, x_2)}
\]  

(5.150)

where \( I \) is the indicator function, not the identity matrix.

**MCX-IPA**

The bicomplex-correlated normal random samples were computed using

\[
[x_k] = \left[\begin{array}{c}
[x_{2,k}]_1 \\
[x_{2,k}]_2
\end{array}\right] \\
= [u_{1,k}I, u_{2,k}I] [L^T] + [[\mu_1], [\mu_2]]
\]  

(5.151)

where \( u_{1,k} \) and \( u_{2,k} \) are independent standard normal samples, \( I \) is a 16 by 16 identity matrix.

The matrix \( [L^T] \) is

\[
[L^T] = \begin{bmatrix}
[L_{11}] & [L_{21}] \\
0I & [L_{22}]
\end{bmatrix}
\]  

(5.152)

where

\[
[L_{11}] = \sigma_1 I \\
[L_{21}] = \rho_{12} \sigma_1 \sigma_2 I / [L_{11}] \\
[L_{22}] = SqrtM([\sigma_2]^2 - [L_{21}]^2)
\]  

(5.153)

The realizations were determined by

\[
[z_k] = [x_{1,k}] - [x_{2,k}]
\]  

(5.154)

where \([z_k]\) is the quadcomplex realization of the response function. The sampling equations that were used to evaluate the multicomplex statistical estimates are given in Section 5.6. And as for the POF sensitivities, recall that the MCX-IPA is not applicable because the indicator function is discontinuous.

**Multicomplex Analytic Approach**
In addition to the MCX-SF and MCX-IPA techniques, multicomplex extensions of the analytic measures were developed, so that higher order multicomplex-step derivatives can be directly obtained from the analytic solutions: bypassing the need to perform a Monte Carlo simulation. Unfortunately, this approach is only applicable for a linear response function with Gaussian random variables.

Since the mean and standard deviation of the response have a closed form solution, the sensitivities of the response moments can be calculated using the analytic expressions as

\[
[\mu_Z] = (\mu_1 - \mu_2)I \quad (5.155)
\]

\[
[\sigma_Z] = \text{SqrtM}(\sigma_1^2 I + [\sigma_2]^2 - 2[\rho_1\sigma_1][\sigma_2]) \quad (5.156)
\]

where \( I \) is the identity matrix.

The POF sensitivities can also be obtained from

\[
[P_f] = \Phi_M([-\beta]) \quad (5.157)
\]

where \( \Phi_M \) is the matrix extension of the standard normal CDF and \([\beta]\) is a multicomplex reliability index.

\[
[\beta] = \frac{[\mu_Z]}{[\sigma_Z]} \quad (5.158)
\]

The matrix standard normal CDF is

\[
\Phi_M([-\beta]) = \frac{1}{2}\left[I + \text{erfM}\left(-[\beta]/\sqrt{2}\right)\right] \quad (5.159)
\]

The matrix error function, \( \text{erfM} \), is

\[
\text{erfM}(A) = I - \frac{I}{\left[I + a_1A + a_2A^2 + a_3A^3 + a_4A^4 + a_5A^5 + a_6A^6\right]^6} \quad (5.160)
\]

where \( a_1, a_2, a_3, a_4, a_5, \) and \( a_6 \) are constants
\[ \begin{align*}
a_1 &= 0.0705230784 \quad a_2 = 0.0422820123 \\
a_3 &= 0.0092705272 \quad a_4 = 0.0001520143 \\
a_5 &= 0.0002765672 \quad a_6 = 0.0000430638 
\end{align*} \] (5.161)

Equation (5.158) is a matrix extension of the rational approximation of the error function given by Abramowitz and Stegun (1964). This error function approximation has an error of \(|\varepsilon(x)| \leq 3 \times 10^{-7}\).

**Results**

Again, let \( T \) denote a generic probabilistic measure. The responses and their derivatives were determined as follows:

\[
T = [T]_{1,1} \tag{5.162}
\]

\[
\frac{\partial T}{\partial \theta_j} = \frac{[T]_{2,1}}{h} = \frac{[T]_{3,1}}{h} = \frac{[T]_{5,1}}{h} = \frac{[T]_{9,1}}{h} \tag{5.163}
\]

\[
\frac{\partial^2 T}{\partial \theta_j^2} = \frac{[T]_{4,1}}{h^2} = \frac{[T]_{6,1}}{h^2} = \frac{[T]_{7,1}}{h^2} = \frac{[T]_{10,1}}{h^2} = \frac{[T]_{11,1}}{h^2} = \frac{[T]_{13,1}}{h^2} \tag{5.164}
\]

\[
\frac{\partial^3 T}{\partial \theta_j^3} = \frac{[T]_{8,1}}{h^3} = \frac{[T]_{12,1}}{h^3} = \frac{[T]_{14,1}}{h^3} = \frac{[T]_{15,1}}{h^3} \tag{5.165}
\]

\[
\frac{\partial^4 T}{\partial \theta_j^4} = \frac{[T]_{16,1}}{h^4} \tag{5.166}
\]

The analytic, MCX-SF, MCX-IPA, and multicomplex analytic results for the sensitivities of the statistical moments of the response are given in Table 5.3 and 5.4.

<table>
<thead>
<tr>
<th>( \theta_j )</th>
<th>( \mu_1 )</th>
<th>( \sigma_1 )</th>
<th>( \mu_2 )</th>
<th>( \sigma_2 )</th>
<th>( \rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial \mu_z}{\partial \theta_j} )</td>
<td>1.42857/</td>
<td>0.00000/</td>
<td>-0.42857/</td>
<td>0.00000/</td>
<td>0.00000/</td>
</tr>
<tr>
<td>( \frac{\partial \theta_j}{\partial \mu_z} )</td>
<td>1.42984/</td>
<td>0.00126/</td>
<td>-0.42863/</td>
<td>-0.00002/</td>
<td>0.0150/</td>
</tr>
</tbody>
</table>
As illustrated in the previous example, the variance of a MCX-SF derivative is greater than the variance of the MCX-IPA derivative for a given number of samples; this is also featured in Table 5.3 and 5.4. Additionally, as shown in tables, the accuracy of the derivative seems to decrease for both methodologies as the order of the derivative increases. This implies that, as the
order of the derivatives increases, a larger number of samples are needed to accurately approximate higher order derivatives using both MCX-SF and MCX-IPA. Figure 5.3 graphically illustrates this behavior by plotting the normalized variance of the derivative of the standard deviation with respect to the correlation coefficient. The variances of the derivatives were calculated from one thousand multicomplex Monte Carlo simulations.

Figure 5.6: Normalized Variance of Higher Order Derivatives

Based on Figure 5.5 one can also conclude that the rate of change (normalized variance/derivative order) for the MCX-SF method is greater than the MCX-IPA technique. For example, the first order derivatives vary by two orders of magnitude in the number of samples. Whereas the fourth order derivatives vary more than two orders of magnitude in the number of samples. The implication of this behavior is that the MCX-IPA technique has a significant
advantage in accurately computing higher order derivatives when compared to the MCX-SF method; fewer MCX-IPA samples are required.

Table 5.5 presents the POF derivative approximations using ten billion samples. Using 8 *labs*, the subroutine took 40 hours to complete.

<table>
<thead>
<tr>
<th></th>
<th>$\mu_1$</th>
<th>$\sigma_1$</th>
<th>$\mu_2$</th>
<th>$\sigma_2$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\partial P_f}{\partial \theta_j}$</td>
<td>-7.45188/</td>
<td>7.58737/</td>
<td>2.23556/</td>
<td>-2.37105/</td>
<td>-4.26789/</td>
</tr>
<tr>
<td>$\frac{\partial^2 P_f}{\partial \theta_j^2}$</td>
<td>47.4196/</td>
<td>27.9063/</td>
<td>4.26776/</td>
<td>3.46433/</td>
<td>5.07867/</td>
</tr>
<tr>
<td>$\frac{\partial^3 P_f}{\partial \theta_j^3}$</td>
<td>-234.013/</td>
<td>-80.5040/</td>
<td>6.31836/</td>
<td>0.99093/</td>
<td>27.7190/</td>
</tr>
<tr>
<td>$\frac{\partial^4 P_f}{\partial \theta_j^4}$</td>
<td>627.229/</td>
<td>-538.687/</td>
<td>5.08056/</td>
<td>-12.1383/</td>
<td>80.4754/</td>
</tr>
</tbody>
</table>

* 10$^8$ samples; analytic/MCX-SF/MCX-analytic
5.8.4. Probabilistic Fracture Mechanics Example

![Diagram of a plate with a circular hole and crack]

Figure 5.7: Single Through Crack in a Plate Emanating From a Circular Hole

The example serves to demonstrate the MCX-SF methodology using a probabilistic fracture mechanics problem presented by Ocampo (2013). Given a particular aircraft component, it is possible that the maximum value of an applied stress, $\sigma_{EVD}$, during flight, can exceed the residual strength, $\sigma_{RS}$, of the component. Thus, the probability of failure at time $t$ is defined as

$$P_f(t) = P[\sigma_{EVD} > \sigma_{RS}]$$

where

$$\sigma_{RS} = \frac{K_c}{Y(a(a_0,t))\sqrt{\pi a(a_0,t)}}$$

The variables $a_0$, $K_c$, and $Y$ denote the initial crack size, fracture toughness, and geometric correction factor of a particular specimen, respectively.
Consider a specimen configuration depicted in Figure 5.6. The problem consists of a plate of width $W$, which contains a circular hole of radius $R$. The plate is subjected to a remote tensile stress $\sigma = \sigma_{EVD}$. There exists a through crack $a$ located on the right edge of the hole. For this crack and geometric configuration, the Mode I stress intensity factor, $K_I$, solution is defined as:

$$K_I = Y \sigma \sqrt{\pi a}$$

(5.169)

where $Y = \beta_{\text{hole}} \beta_{\text{width}}$ is known as the geometric correction factor. The variables $\beta_{\text{hole}}$ and $\beta_{\text{width}}$ are

$$\beta_{\text{hole}} = 0.6762 + \frac{0.8734}{0.3246 + \frac{a}{R}}$$

(5.170)

and

$$\beta_{\text{width}} = \sqrt{\sec \left( \frac{\pi (R + a)}{W} \right)}.$$  

(5.171)

Working on the assumption that the crack growth rate, which is defined as the slope of the $a$ versus $t$ curve, is expressed as $\frac{da}{dt} = Qa$. The crack size $a$ at time $t$ is then defined by the following exponential function:

$$a(t) = a_0 \exp(Qt)$$

(5.172)

where $a(t)$ is the crack size at time $t$ and $Q$ is a parameter that has been fitted to experimental crack growth data.

In this example, three random variables are considered: a Gumbel-distributed maximum applied stress $\sigma_{EVD}$, a lognormally distributed initial crack size $a_0$, and a normally distributed
fracture toughness $K_c$. The problem parameters are given in Table 1. The PDF for the initial crack size and fracture toughness are given in Figures 5.7 and Figure 5.8.

### Table 5.6: Fracture Mechanics Parameter Values

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
<th>Natural Parameters</th>
<th>Gumbel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius, $R$</td>
<td>0.125 in.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Width, $W$</td>
<td>1,000 in.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exponential growth parameter, $Q$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Maximum applied stress, $\sigma_{EVD}$</td>
<td></td>
<td>Gumbel</td>
<td></td>
</tr>
<tr>
<td>Natural Parameters</td>
<td>Location $= a = 14.5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Scale $= b = 0.8$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Moments</td>
<td>$\mu_{\sigma_{EVD}} = 14.9617$ ksi</td>
<td></td>
<td>$\sigma_{\sigma_{EVD}} = 1.02603$ ksi</td>
</tr>
<tr>
<td>Initial crack size, $a_0$</td>
<td>Lognormal</td>
<td>$\lambda = -6.420$</td>
<td></td>
</tr>
<tr>
<td>Natural Parameters</td>
<td>$\zeta = 1.113$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Moments</td>
<td>$\mu_{a_0} = 0.0030257$ in.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\sigma_{a_0} = 0.0047373$ in.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fracture Toughness, $K_c$</td>
<td>Normal</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\mu_{K_c} = 34.8$ ksi $\sqrt{\text{in}}$</td>
<td></td>
<td>$\sigma_{K_c} = 3.9$ ksi $\sqrt{\text{in}}$</td>
</tr>
</tbody>
</table>

The probability of failure in terms of the indicator function is given by

$$P_f(t) = \int_{-\infty}^{\infty} I(\sigma_{EVD}, K_c, da_0) \cdot f_{\sigma_{EVD}}(\sigma_{EVD}) \cdot f_{K_c}(K_c) \cdot f_{a_0}(a_0) d\sigma_{EVD} dK_c da_0$$

(5.173)

Using conditional expectation, the probability of failure can be expressed in terms of the CDF of the maximum stress $F_{EVD}$ as

$$P_f(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ 1 - F_{EVD} \left( \frac{K_c}{Y(a_0, t) \sqrt{\pi a_0(t)}} \right) \right] f_{a_0}(a_0) f_{K_c}(K_c) da_0 dK_c$$

(5.174)
A sensitivity analysis of the probability of failure with respect to several statistical parameters at $t=12,000$ was conducted using the MCX-Gauss quadrature rule, MCX-SF, and MCX-IPA technique. In this problem, the MCX-IPA technique is applicable because the POF is not written in terms of the indicator function. In particular, first and second order derivatives were computed using bicomplex numbers. The first and second order normalized derivatives of the POF with respect to the moments of the initial crack size and fracture toughness are given in Table 5.7 and Table 5.8.

Table 5.7: Comparison of Higher Order Sensitivities for the Fracture Mechanics Example

<table>
<thead>
<tr>
<th>$\theta_i$</th>
<th>$\mu_{\sigma_i}$</th>
<th>$\sigma_{\sigma_i}$</th>
<th>$\mu_{K_c}$</th>
<th>$\sigma_{K_c}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\partial P_f(12,000) \sigma_i / \partial \theta_i P_f$</td>
<td>-2.39808/</td>
<td>4.80567/</td>
<td>-0.95702/</td>
<td>1.02682/</td>
</tr>
<tr>
<td></td>
<td>-2.41860/</td>
<td>4.83633/</td>
<td>-0.97699/</td>
<td>1.05179/</td>
</tr>
<tr>
<td>$\partial^2 P_f(12,000) \sigma_i^2 / \partial \theta_i^2 P_f$</td>
<td>3.07543/</td>
<td>10.9895/</td>
<td>1.02682/</td>
<td>2.72902/</td>
</tr>
<tr>
<td></td>
<td>3.04891/</td>
<td>10.5523</td>
<td>1.06356/</td>
<td>2.76695/</td>
</tr>
<tr>
<td></td>
<td>3.15499</td>
<td>10.0841</td>
<td>1.03837</td>
<td>2.94621</td>
</tr>
</tbody>
</table>

*10^7 samples; MCX-Gauss Quadrature / MCX-SF / MCX-IPA

Table 5.8: Comparison of Second Order Mixed Sensitivities for the Fracture Mechanics Example

<table>
<thead>
<tr>
<th>$\theta_i$</th>
<th>$\mu_{\sigma_i}$</th>
<th>$\sigma_{\sigma_i}$</th>
<th>$\mu_{K_c}$</th>
<th>$\sigma_{K_c}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_j$</td>
<td>$\mu_{\sigma_i}$</td>
<td>-</td>
<td>-</td>
<td>1.19084/</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-</td>
<td>-</td>
<td>1.09057</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-</td>
<td>-</td>
<td>1.24584</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{\sigma_i}$</td>
<td>-</td>
<td>-</td>
<td>-3.64567/</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-</td>
<td>-</td>
<td>-3.46478</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-</td>
<td>-</td>
<td>-3.33816</td>
</tr>
<tr>
<td></td>
<td>$\mu_{K_c}$</td>
<td>-</td>
<td>-</td>
<td>-1.24089/</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-</td>
<td>-</td>
<td>-1.24089/</td>
</tr>
</tbody>
</table>

*10^7 Samples; MCX-Gauss Quadrature / MCX-SF / MCX-IPA
MCX-Gauss Quadrature

The MCX-Gauss quadrature derivatives were calculated using a 50-point Gaussian quadrature rule in 2D (i.e., $M=N=50$). That is, 2,500 quadrature points were used within the 2D square domain. The quadrature points, $x_i$ and $x_j$, as well as the weights, $w_i$ and $w_j$, were calculated using a MATLAB code given by Trefethen (2008). This code allows the user to specify any number of quadrature points. The finite limits of integration for both random variables were determined graphically from their respected PDF; values that were several standard deviations away from the mean were chosen. The finite upper and lower limits of integration of initial crack size were $a = 10^{-7}$ in. and $b = 1$ in. The limits of integration of the fracture toughness were $c = 10$ ksi and $d = 60$ ksi.

The 2D Gaussian quadrature integral of the probability of failure is given as

$$P_f(t) = \frac{b-a}{2} \int_{-1}^{1} \int_{-1}^{1} \left[ 1 - F_{EVD} \left( \frac{K_c^*}{Y(a(a_0^*,t))\sqrt{\pi}a(a_0^*,t)} \right) \right] \left[ f_{a_0}(a_0^*) \right] \left[ f_{K_c}(K_c^*) \right] da_0 dK_c \approx \frac{b-a}{2} \sum_{j=1}^{N} \sum_{i=1}^{M} w_i w_j \left[ 1 - F_{EVD} \left( \frac{K_c^*}{Y(a(a_0^*,t))\sqrt{\pi}a(a_0^*,t)} \right) \right] \left[ f_{a_0}(a_0^*) \right] \left[ f_{K_c}(K_c^*) \right]$$

(5.175)

where $a_0^*$ and $K_c^*$ are

$$a_0^* = \frac{b-a}{2} x_j + \frac{b+a}{2}$$

(5.176)

$$K_c^* = \frac{d-c}{2} x_j + \frac{d+c}{2}$$

(5.177)

Prior to evaluating Eq. (5.173), an appropriate bicomplex perturbation was applied to a moment parameter of interest. Note that only the PDF's, $[f_{a_0}(a_0^*)]$ and $[f_{K_c}(K_c^*)]$, were expanded to their matrix form. The first and second order derivatives of the probability of failure with respect to
the moment parameters of the initial crack size and fracture toughness are given in Table 5.6 and Table 5.7.

**Discussion**

The development of higher order statistical estimates can of course be generated analytically by repeatedly differentiating the governing probabilistic equations. As the derivative order increases the number of terms in the derivative expression also increase due to the product rule. In addition, higher order derivatives use higher order kernel functions (SF method) and higher order IPA estimators. However, the multicomplex extensions of these methodologies have some significant advantages: simplicity, extensibility, and adaptability.

The numerical techniques are simple in concept and programming once the needed matrix functions are in place. No additional effort is needed to consider any order derivative including mixed partials; the size of the perturbation matrix and the location of the perturbed step size within the matrix controls the derivatives to be computed. Therefore, the user need only program the matrix PDF function (MCX-SF) or the matrix response function (MCX-IPA).

In contrast, an analytical development requires a lengthy number of kernel functions and IPA estimators to be programmed correctly for each distribution under consideration, including mixed partial derivatives. For example, the bivariate normal contains 5 parameters \((\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)\), hence 5 first order derivatives are needed, 15 second order terms, 15 third order terms, etc.

Lastly, both methodologies extend easily to any order. Once the code framework has been established, one merely changes the order of complex perturbation (e.g., size of the Cauchy-Riemann matrix) to obtain the desired derivatives. Hence if the matrix functions are
available (e.g., exponential, log, power, square root) the distribution or response can be added easily.
APPENDIX

Appendix A. Multicomplex Matrix

The following MATLAB function populates an equivalent matrix representation of a multicomplex number relevant to the MCX method.

```matlab
function [F] = GMM(comp_order,dir,h,val)
% [F] = GMM(comp_order,dir,h,val) generates a multicomplex matrix with
% appropriate perturbations
%
% INPUT ARGUMENTS:
% comp_order - highest order derivative to be computed (e.g., 1, 2, 3, etc.)
% h - step size
% val - the real part of the multicomplex number
% dir - the imaginary direction(s) that is/are perturbed
% (e.g., dir = [1,2,4] adds a perturbation to the i1, i2, and i4 direction; dir='r' returns val times the identity matrix)
%
C_matrix = zeros(2^comp_order);
leng = length(C_matrix);

if dir=='r'
    F = diag(val.*ones(leng,1));
else
    for i=dir
        bi_coeff = 2^(i-1);
leng_x = leng - bi_coeff;
x = zeros(leng_x,1);
        for j=1:bi_coeff
            x(j:2^i:end) = h;
        end
        C_matrix = diag(-x,bi_coeff) + diag(x,-bi_coeff) + C_matrix;
    end

    F = val.*diag(ones(leng,1)) + C_matrix;
end
```
Appendix B. Analytical Derivatives of Statistical Estimates

Mean:

\[
\frac{\partial \mu_z}{\partial \theta_1} = \int_{-\infty}^{\infty} z(x) \cdot \frac{\partial^2 f_x(x)}{\partial \theta_1} \, dx
\]

(A.1)

\[
\frac{\partial^2 \mu_z}{\partial \theta_1 \partial \theta_2} = \int_{-\infty}^{\infty} z(x) \cdot \frac{\partial^2 f_x(x)}{\partial \theta_1 \partial \theta_2} \, dx
\]

(A.2)

\[
\frac{\partial^3 \mu_z}{\partial \theta_1 \partial \theta_2 \partial \theta_3} = \int_{-\infty}^{\infty} z(x) \cdot \frac{\partial^3 f_x(x)}{\partial \theta_1 \partial \theta_2 \partial \theta_3} \, dx
\]

(A.3)

\[
\frac{\partial^4 \mu_z}{\partial \theta_1 \partial \theta_2 \partial \theta_3 \partial \theta_4} = \int_{-\infty}^{\infty} z(x) \cdot \frac{\partial^4 f_x(x)}{\partial \theta_1 \partial \theta_2 \partial \theta_3 \partial \theta_4} \, dx
\]

(A.4)

Variance:

\[
\frac{\partial V_z}{\partial \theta_1} = \int_{-\infty}^{\infty} z(x)^2 \cdot \frac{\partial f_x(x)}{\partial \theta_1} \, dx - 2 \mu_z \frac{\partial \mu_z}{\partial \theta_1}
\]

(A.5)

\[
\frac{\partial^2 V_z}{\partial \theta_1 \partial \theta_2} = \int_{-\infty}^{\infty} z(x)^2 \cdot \frac{\partial^2 f_x(x)}{\partial \theta_1 \partial \theta_2} \, dx - 2 \left[ \frac{\partial \mu_z}{\partial \theta_1} \frac{\partial \mu_z}{\partial \theta_2} + \mu_z \frac{\partial^2 \mu_z}{\partial \theta_1 \partial \theta_2} \right]
\]

(A.6)

\[
\frac{\partial^3 V_z}{\partial \theta_1 \partial \theta_2 \partial \theta_3} = \int_{-\infty}^{\infty} z(x)^2 \cdot \frac{\partial^3 f_x(x)}{\partial \theta_1 \partial \theta_2 \partial \theta_3} \, dx -
\]
\[
2 \left[ \frac{\partial^3 \mu_z}{\partial \theta_1 \partial \theta_2 \partial \theta_3} \frac{\partial \mu_z}{\partial \theta_3} + \frac{\partial \mu_z}{\partial \theta_1} \frac{\partial^3 \mu_z}{\partial \theta_2 \partial \theta_3} + \frac{\partial \mu_z}{\partial \theta_2} \frac{\partial^3 \mu_z}{\partial \theta_3} + \mu_z \frac{\partial^3 \mu_z}{\partial \theta_1 \partial \theta_2 \partial \theta_3} \right]
\]

(A.7)

\[
\frac{\partial^4 V_z}{\partial \theta_1 \partial \theta_2 \partial \theta_3 \partial \theta_4} = \int_{-\infty}^{\infty} z(x)^2 \cdot \frac{\partial^4 f_x(x)}{\partial \theta_1 \partial \theta_2 \partial \theta_3 \partial \theta_4} \, dx -
\]
\[
2 \left[ \frac{\partial^3 \mu_z}{\partial \theta_1 \partial \theta_2 \partial \theta_3 \partial \theta_4} \frac{\partial \mu_z}{\partial \theta_4} + \frac{\partial \mu_z}{\partial \theta_1} \frac{\partial^3 \mu_z}{\partial \theta_2 \partial \theta_3 \partial \theta_4} + \frac{\partial \mu_z}{\partial \theta_2} \frac{\partial^3 \mu_z}{\partial \theta_3 \partial \theta_4} + \frac{\partial \mu_z}{\partial \theta_3} \frac{\partial^3 \mu_z}{\partial \theta_4} + \mu_z \frac{\partial^4 \mu_z}{\partial \theta_1 \partial \theta_2 \partial \theta_3 \partial \theta_4} \right]
\]

(A.8)
Standard Deviation:

\[
\frac{\partial \sigma_Z}{\partial \theta_1} = \frac{1}{2\sigma_Z} \frac{\partial V_Z}{\partial \theta_1} \tag{A.9}
\]

\[
\frac{\partial^2 \sigma_Z}{\partial \theta_1 \partial \theta_2} = \frac{1}{2} \left[ \frac{\partial^2 V_Z}{\partial \theta_1 \partial \theta_2} - \frac{1}{\sigma_Z^2} \frac{\partial \sigma_Z}{\partial \theta_1} \frac{\partial \sigma_Z}{\partial \theta_2} \right] \tag{A.10}
\]

\[
\frac{\partial^3 \sigma_Z}{\partial \theta_1 \partial \theta_2 \partial \theta_3} = \frac{1}{2} \left[ \frac{\partial^3 V_Z}{\partial \theta_1 \partial \theta_2 \partial \theta_3} - \frac{1}{\sigma_Z^2} \frac{\partial \sigma_Z}{\partial \theta_1} \frac{\partial^2 \sigma_Z}{\partial \theta_2 \partial \theta_3} + \frac{2}{\sigma_Z^3} \frac{\partial \sigma_Z}{\partial \theta_1} \frac{\partial \sigma_Z}{\partial \theta_2} \frac{\partial \sigma_Z}{\partial \theta_3} \right] \tag{A.11}
\]

\[
\frac{\partial^4 \sigma_Z}{\partial \theta_1 \partial \theta_2 \partial \theta_3 \partial \theta_4} = \frac{1}{2} \left[ \frac{\partial^4 V_Z}{\partial \theta_1 \partial \theta_2 \partial \theta_3 \partial \theta_4} - \frac{1}{\sigma_Z^2} \frac{\partial \sigma_Z}{\partial \theta_1} \frac{\partial^3 \sigma_Z}{\partial \theta_2 \partial \theta_3 \partial \theta_4} + \frac{2}{\sigma_Z^3} \frac{\partial \sigma_Z}{\partial \theta_1} \frac{\partial \sigma_Z}{\partial \theta_2} \frac{\partial^2 \sigma_Z}{\partial \theta_3 \partial \theta_4} + \frac{1}{\sigma_Z^4} \frac{\partial \sigma_Z}{\partial \theta_1} \frac{\partial \sigma_Z}{\partial \theta_2} \frac{\partial \sigma_Z}{\partial \theta_3} \frac{\partial \sigma_Z}{\partial \theta_4} \right] \tag{A.12}
\]
REFERENCES


