A Derivation of Bill James' Pythagorean Won-Loss Formula

These notes were compiled by John Paul Cook from a paper by Dr. Stephen J. Miller, an Assistant Professor of Mathematics at Williams College, for a talk given to the Applied Mathematics Seminar at the University of Oklahoma on April 17, 2009.

1. The Pythagorean Formula

Sabermetrician Bill James derived the Pythagorean W-L Formula to serve as an expected value for a baseball team's winning percentage, based on the number of runs a team scores and allows over the course of a season:

$$\text{Expected Win Percentage} = \frac{\text{Runs Scored}^2}{\text{Runs Scored}^2 + \text{Runs Allowed}^2}$$

By James' own admission, his own derivation of the formula, while the result of data analysis, had little basis in statistical and mathematical theory. We will show that, under reasonable statistical assumptions, James' formula does, in fact, follow mathematically.

Example 1.1. The application of this formula:

(1) On August 4, 2008 the Texas Rangers had scored 640 runs and allowed 667, leading to a Pythagorean W-L% of .478 (54-59). Their actual record at the time was .522 (59-54). The Rangers ended the season at .488 (79-83).

(2) On July 20, 2008 the Cleveland Indians had scored 443 runs and allowed 434, leading to a Pythagorean W-L% of .505 (49-48). Their actual record at the time was .443 (43-54). The Indians ended the season at .500 (81-81).

2. Assumptions and the Weibull Distribution

Remark 2.1. We will be making the following assumptions:

(1) Runs scored and runs allowed can be modeled by continuous random variables
- This allows us to replace discrete sums with continuous integrals, which are much easier to solve and much nicer to work with.
- While continuous run distributions do not make sense, we hope that this computationally useful assumption reasonably approximates the corresponding discrete distribution.

(2) Runs scored and runs allowed can be modeled by three-parameter Weibull distributions
- Weibull random variables have flexible shape parameters which make it easier to fit to observed data than better known distributions, such as the exponential.
- The exponential distribution decays too slowly to be realistic, leading to too many games with absurdly high scores, while the decay of the Weibull distribution is much more realistic.

(3) Runs scored and runs allowed per game are statistically independent
- While they can not be entirely independent (as games can not end in ties), modified Chi-Squared tests have shown that runs scored and runs allowed per game are statistically independent.

Definition 2.2. The probability density function for a **three-parameter Weibull distribution** is given by:

$$f(x;\alpha,\beta,\gamma) = \left(\frac{\gamma}{\alpha}\right)\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1} e^{-\left(\frac{x-\beta}{\alpha}\right)^{\gamma}} \text{ if } x \geq \beta, \text{ 0 otherwise}$$

Lemma 2.3. The expected value for a three-parameter Weibull random variable X is given by

$$E(X) = \alpha\Gamma(1+\gamma^{-1}) + \beta$$

where Γ is the Γ-function

$$\Gamma(s) = \int_0^\infty e^{-u}u^{s-1}\, du \text{ for } s \in \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$$

Example 2.4. Using the Weibull distribution:
Suppose that the Texas Rangers' runs scored per game is given by a Weibull distribution with parameters $(\alpha,\beta,\gamma) = (5,0,2)$. Then the probability that a team scores between 1 and 3 runs in a given game is

$$\int_1^3 \left(\frac{2}{5}\right)\left(\frac{x}{5}\right)e^{-\frac{x^2}{5}}\, dx = .27$$

Remark 2.5. Some notes on the parameters:

(1) γ is a generalized shape parameter. When $\gamma = 1$ we have the exponential distribution, and when $\gamma = 2$ we have the Rayleigh distribution. The γ parameter will end up being the exponent in the final formula. While Bill James used $\gamma = 2$ (most likely for simplicity), subsequent analysis of James' formula has shown a more accurate exponent to be about 1.82. We generalize

this parameter to leave open the possibility to find a best-fit exponent for whatever league is being analyzed. Analysis of the data has shown that we can get good fits for runs scored and runs allowed by using the same γ for both runs scored and runs allowed.

(2) β is incorporated to partially account for our use of continuous random variables instead of discrete random variables. It will also help adjust this formula to other sports by allowing us to examine scores above a baseline (for example, no basketball team scores fewer than 25 points in a game). Incorporation of this parameter will give us a final result of:

$$\text{Expected Win Percentage} = \frac{(RS - \beta)^{\gamma}}{(RS - \beta)^{\gamma} + (RA - \beta)^{\gamma}}$$

We use the same β for both runs scored and runs allowed because adding the same number to both runs scored and runs allowed will not change the outcome of the game, and thus preserves the expected winning percentage.

(3) Recall from above that if X is a Weibull RV,

$$E(X) = \alpha\Gamma(1 + \gamma^{-1}) + \beta$$

Since we are using the same γ and β for runs scored and runs allowed, we will use distinct α for both (which we denote α_{RS} and α_{RA}). In fact, we will choose α_{RS} and α_{RA} such that the random variables for runs scored and runs allowed conform to the above expected value formula.

We are almost ready to prove our main result, but we need a few more preliminaries. Recall that our goal is to calculate the projected winning percentage of a baseball team. In other words, the probability that, in any given game, the team in question will score more runs than it allows. Since this probability depends *jointly* on our two (Weibull) random variables for runs scored and runs allowed, we need to use a joint probability density function:

Definition 2.6. Suppose that X and Y are two continuous random variables defined over a given sample space S. The **joint probability density function** of X and Y, $f_{X,Y}(x,y)$, is the surface having the property that for any region R in the xy-plane,

$$P(X, Y \in R) = \int\int_R f_{X,Y}(x,y)\, dx\, dy$$

Additionally, as we discussed earlier, we are assuming runs scored and runs allowed to be statistically independent:

Definition 2.7. Two random variables X and Y are (statistically) independent if and only if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) \text{ for all } x, y.$$

Remark 2.8. Also recall that the integral over all possible values of a probability density function is 1:

$$\int_{-\infty}^{\infty} f_X(x)\ dx = 1$$

We are now ready to proceed.

3. Proof of Bill James' Pythagorean Won-Loss Formula

Theorem 3.1. Let runs scored and runs allowed be independent three-parameter Weibull random variables denoted by $(X, \alpha_{RS}, \beta, \gamma)$ and $(Y, \alpha_{RA}, \beta, \gamma)$, respectively. Suppose α_{RS} and α_{RA} are chosen such that $E(X) = \alpha_{RS}\Gamma(1+\gamma^{-1}) + \beta$ and $E(Y) = \alpha_{RA}\Gamma(1+\gamma^{-1}) + \beta$. Put $E(X) = RS$ and $E(Y) = RA$. If $\gamma > 0$, then

$$\text{Expected Win Percentage} = P(X > Y) = \frac{(RS-\beta)^\gamma}{(RS-\beta)^\gamma + (RA-\beta)^\gamma}$$

Proof. First of all, we have that

$$RS = \alpha_{RS}\Gamma(1+\gamma^{-1}) + \beta \text{ and } RA = \alpha_{RA}\Gamma(1+\gamma^{-1}) + \beta$$

So that

$$\alpha_{RS} = \frac{RS-\beta}{\Gamma(1+\gamma^{-1})} \text{ and } \alpha_{RA} = \frac{RA-\beta}{\Gamma(1+\gamma^{-1})}$$

Now

$P(X > Y)$

$$= \int_{x=\beta}^{\infty}\int_{y=\beta}^{x} f_{X,Y}(x,y)\ dy\ dx = \int_{x=\beta}^{\infty}\int_{y=\beta}^{x} f_X(x) f_Y(y)\ dy\ dx$$

$$= \int_{x=\beta}^{\infty}\int_{y=\beta}^{x} \frac{\gamma}{\alpha_{RS}}\left(\frac{x-\beta}{\alpha_{RS}}\right)^{\gamma-1} e^{-\left(\frac{x-\beta}{\alpha_{RS}}\right)^{\gamma}} \frac{\gamma}{\alpha_{RA}}\left(\frac{y-\beta}{\alpha_{RA}}\right)^{\gamma-1} e^{-\left(\frac{y-\beta}{\alpha_{RA}}\right)^{\gamma}}\ dy\ dx$$

$$= \int_{x=0}^{\infty}\int_{y=0}^{x} \frac{\gamma}{\alpha_{RS}}\left(\frac{x}{\alpha_{RS}}\right)^{\gamma-1} e^{-\left(\frac{x}{\alpha_{RS}}\right)^{\gamma}} \frac{\gamma}{\alpha_{RA}}\left(\frac{y}{\alpha_{RA}}\right)^{\gamma-1} e^{-\left(\frac{y}{\alpha_{RA}}\right)^{\gamma}}\ dy\ dx$$

$$= \int_{x=0}^{\infty} \frac{\gamma}{\alpha_{RS}}\left(\frac{x}{\alpha_{RS}}\right)^{\gamma-1} e^{-(x/\alpha_{RS})^{\gamma}} \int_{y=0}^{x} \frac{\gamma}{\alpha_{RS}}\left(\frac{y}{\alpha_{RA}}\right)^{\gamma-1} e^{-(y/\alpha_{RA})^{\gamma}}\ dy\ dx$$

$$= \int_{x=0}^{\infty} \frac{\gamma}{\alpha_{RS}}\left(\frac{x}{\alpha_{RS}}\right)^{\gamma-1} e^{-(x/\alpha_{RS})^{\gamma}} \left[e^{-(y/\alpha_{RA})}\right]_{y=0}^{y=x}\ dx$$

$$= \int_{x=0}^{\infty} \frac{\gamma}{\alpha_{RS}}\left(\frac{x}{\alpha_{RS}}\right)^{\gamma-1} e^{-(x/\alpha_{RS})^{\gamma}} \left[1 - e^{-(x/\alpha_{RS})}\right]\ dx$$

$$= \int_{x=0}^{\infty} \frac{\gamma}{\alpha_{RS}}\left(\frac{x}{\alpha_{RS}}\right)^{\gamma-1} e^{-(x/\alpha_{RS})^{\gamma}} - \frac{\gamma}{\alpha_{RS}}\left(\frac{x}{\alpha_{RS}}\right)^{\gamma-1} e^{\left(-x\left(\frac{1}{\alpha_{RS}^{\gamma}}+\frac{1}{\alpha_{RA}^{\gamma}}\right)\right)}\ dx$$

$$= 1 - \int_{x=0}^{\infty} \frac{\gamma}{\alpha_{RS}}\left(\frac{x}{\alpha_{RS}}\right)^{\gamma-1} e^{\left(-x\left(\frac{1}{\alpha_{RS}^{\gamma}}+\frac{1}{\alpha_{RA}^{\gamma}}\right)\right)}\ dx$$

Let $\alpha^{\gamma} = \dfrac{\alpha_{RS}^{\gamma}\alpha_{RA}^{\gamma}}{\alpha_{RS}^{\gamma}+\alpha_{RA}^{\gamma}}$. Then multiplying the previous expression by $\dfrac{\alpha^{\gamma}}{\alpha^{\gamma}}$ gives us:

$$= 1 - \frac{\alpha_{RA}^{\gamma}}{\alpha_{RS}^{\gamma}+\alpha_{RA}^{\gamma}} \int_{x=0}^{\infty} \frac{\gamma}{\alpha}\left(\frac{x}{\alpha}\right)^{\gamma-1} e^{\left(\frac{x}{\alpha}\right)^{\gamma}}\ dx$$

$$= 1 - \frac{\alpha_{RA}^{\gamma}}{\alpha_{RS}^{\gamma}+\alpha_{RA}^{\gamma}}$$

$$= \frac{\alpha_{RS}^{\gamma}}{\alpha_{RS}^{\gamma}+\alpha_{RA}^{\gamma}}$$

$$= \frac{\left(\frac{RS-\beta}{\Gamma(1+\gamma^{-1})}\right)^{\gamma}}{\left(\frac{RS-\beta}{\Gamma(1+\gamma^{-1})}\right)^{\gamma} + \left(\frac{RA-\beta}{\Gamma(1+\gamma^{-1})}\right)^{\gamma}}$$

$$= \frac{(RS-\beta)^{\gamma}}{(RS-\beta)^{\gamma}+(RA-\beta)^{\gamma}}$$

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