Trees and Amalgamated Products

Notes on *Trees* by Jean-Pierre Serre

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1. Group Actions on Trees

Definition 1.1. A **group** G **acts on a graph** X, denoted by $G \times X \to X$, if G acts on the vertices and edges of X:

(1) $G \times \mathrm{vert}(X) \to \mathrm{vert}(X)$
(2) $G \times \mathrm{edge}(X) \to \mathrm{edge}(X)$

and the action commutes with the usual incidence functions $o, t : \mathrm{edge}(X) \to \mathrm{vert}(X)$:

$$o(gy) = g(o(y))$$
$$t(gy) = g(t(y))$$

where $g \in G$, $x \in \mathrm{vert}(X)$, and $y \in \mathrm{edge}(X)$.

Definition 1.2. Let G be a group and X a graph upon which G acts.

a. An **inversion** is a pair consisting of some $g \in G$ and an edge y of X such that $gy = \overline{y}$ (where $\overline{y}$ is the reverse edge of y).
b. If no such pair exists we say that G acts **without inversion** on X. In other words, the action does not map any edge to its reverse edge (and thus preserves the orientation of X).

If G acts on X without inversion, then we can define the quotient graph $G \backslash X$ (read: X mod G) in an obvious way:

- The vertex set of $G \backslash X$ is the quotient of $\mathrm{vert}(X)$ under the action of G: $\mathrm{vert}(G \backslash X) = \{Gx : x \in \mathrm{vert}(X)\}$
- The edge set of $G \backslash X$ is the quotient of $\mathrm{edge}(X)$ under the action of G: $\mathrm{edge}(G \backslash X) = \{Gy : y \in \mathrm{edge}(X)\}$

Definition 1.3. A **tree** is a connected, nonempty graph without circuits.

Definition 1.4. Let G be a group acting on a tree X without inversion. A **fundamental domain** of $G \backslash X$ is a subgraph T of X such that $T \to G \backslash X$ is an isomorphism.

Proposition 1.5. Let G be a group acting upon a tree X without inversion. Then every subtree T' of $G \backslash X$ lifts to a subtree T of X.

Proposition 1.6. Let G be a group acting without inversion on a tree X. A fundamental domain of $G \backslash X$ exists if and only if $G \backslash X$ is a tree.

Proof. ($\Rightarrow$) Let T be a fundamental domain of $G \backslash X$. Since X is connected and non-empty, then $G \backslash X \cong T$ is connected and non-empty. So T is a tree as a non-empty, connected subgraph of a tree. Thus $G \backslash X$ is a tree.
($\Leftarrow$) Suppose $G \backslash X$ is a tree. By Proposition 1.5, $G \backslash X$ is isomorphic to a subtree of X, call it T. This is the desired fundamental domain. $\square$

2. Free Products with Amalgamated Subgroups

Definition 2.1. Suppose the $G = \langle S_G | R_G \rangle$ is a presentation of G where S_G is a set of generators and R_G is a set of relations. Similarly, suppose that $H = \langle S_H | R_H \rangle$ is a presentation of H. Then the **free product of G and H**, denoted $G * H$, is given by:

$$G * H = \langle S_G \cup S_H | R_G \cup R_H \rangle$$

Definition 2.2. Suppose that G and H are as defined above and contain an isomorphic copy of the group F. Let $i_G : F \hookrightarrow G$ and $i_H : F \hookrightarrow H$ be the respective inclusions. Let $R_F = \{i_G(f)i_H(f)^{-1} : f \in F\}$. Then the **free product of *G* and *H* with amalgamated subgroup *F*** (also called the **amalgam of *G* and *H* over *F***), denoted $G *_F H$, is given by:

$$G *_F H = \langle S_G \cup S_H | R_G \cup R_H \cup R_F \rangle$$

Example 2.3. Take

$$\begin{aligned} G &= \mathbb{Z}/4\mathbb{Z} = \langle a | a^4 \rangle \\ H &= \mathbb{Z}/6\mathbb{Z} = \langle b | b^6 \rangle \\ F &= \mathbb{Z}/2\mathbb{Z} = \langle c | c^2 \rangle \end{aligned}$$

along with the inclusions:

$$\begin{aligned} i_G &: \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/4\mathbb{Z}, \text{ where } i_G(\mathbb{Z}/2\mathbb{Z}) = \{0, 2\} \subset \mathbb{Z}/4\mathbb{Z} \\ i_H &: \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/6\mathbb{Z}, \text{ where } i_H(\mathbb{Z}/2\mathbb{Z}) = \{0, 3\} \subset \mathbb{Z}/6\mathbb{Z} \end{aligned}$$

So we have that $\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} = \langle a, b \big| a^4, b^6, i_G(c)i_H(c)^{-1} \rangle$.

3. Trees and Amalgams

As it turns out, every group that acts without inversion on a graph with a segment as fundamental domain is an amalgam of two groups, and the graph is a tree:

Theorem 3.1. Let G be a group acting without inversion on a graph X, and let T be a segment of X that has edge y (reverse edge $\overline{y}$) with $o(y) = P$, $t(y) = Q$. Suppose that T is a fundamental domain of $G\backslash X$. Let G_P, G_Q, and $G_y = G_{\overline{y}}$ be the respective stabilizers of y, P, and Q under the action of G. Then X is a tree if and only if the homomorphism $G_P *_{G_y} G_Q \to G$ induced by the inclusions $G_P \to G$ and $G_Q \to G$ is an isomorphism. *Note*: This amalgam makes sense because $G_P \cap G_Q = G_y$.

Proof. We shall need the following two lemmas:

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Lemma 3.2. X is connected if and only if G is generated by $G_P \cup G_Q$.

Proof. Let X' be the component of X containing T, and let G' be the set of elements that stabilize X', i.e. $G' = \{g \in G : gX' = X'\}$. Let G'' be the subgroup of G generated by $G_P \cup G_Q$.
We want to show that $G = G''$. Let $h \in G_P \cup G_Q$. Then T and hT share a common vertex, which gives that $hT \subset X' \Rightarrow hX' = X' \Rightarrow h \in G'$. Thus $G'' \subset G'$.
On the other hand, notice that $G''T$ and $(G - G'')T$ are disjoint subgraphs of X whose union is X. So either $X' \subset G''T$ or $X' \subset (G - G'')T$. Since $T = 1_{G''}T \in G''T$ then $X' \subset G''T$. Thus $G' \subset G'' \Rightarrow G = G' = G''$.

Now $X = X'$ (i.e. X is connected) if and only if $G = G' = G''$. □

Lemma 3.3. X contains no circuit if and only if $G_P *_{G_y} G_Q \to G$ is injective.

Proof. We know that X contains a circuit if and only if there is a path $c = (w_0, \cdots, w_n)$, $n \geq 1$, in X without backtracking such that $o(c) = t(c)$. Write $w_i = h_i y_i$, where $h_i \in G$ and $y_i = y$ or $\overline{y}$. Passing to the quotient $G\backslash X \cong T$ we get that $\overline{y_i} = y_{i-1}$. Let $P_i = o(y_i) = t(y_{i-1})$. Notice that:

$$h_i P_i = h_i o(y_i) = o(h_i y_i) = t(h_{i-1} y_{i-1}) = h_{i-1} t(y_{i-1}) = h_{i-1} P_i$$

So $g_i \in G_{P_i}$ where $h_i = h_{i-1} g_i$. Also notice that $\overline{h_i y_i} \neq h_{i-1} y_{i-1}$ so that $g_i \notin G_y$. Notice that $o(c) = t(c)$ is equivalent to writing $t(y_n) = P_0$, which is also equivalent to:

$$h_0 P_0 = h_n P_0 = h_{n-1} g_n P_0 = h_{n-2} g_{n-1} g_n P_0 = \ldots = h_0 g_1 g_2 \cdots g_n P_0$$

i.e. $g_1 g_2 \cdots g_n \in G_{P_0}$.

Thus, X contains a circuit if and only if we can find a sequence of vertices of T with $\{P_{i-1}, P_i\} = \{P, Q\}$ for all i and a sequence of elements $g_i \in G_{P_i} - G_y$ $(0 \leq i \leq n)$, such that $g_0 g_1 \cdots g_n = 1$. So $G_P *_{G_y} G_Q \to G$ is not injective. □

These two lemmas together form the statement: X is a tree if and only if $G_P *_{G_y} G_Q \to G$ is an isomorphism. □

The converse is also true: every amalgam of two groups acts on a tree with a segment as fundamental domain:

Theorem 3.4. Let $G = G_1 *_A G_2$ be the amalgam of G_1 and G_2 over A. Then there is a tree X (unique up to isomorphism) on which G acts, with the segment T (as defined in the previous theorem) as fundamental domain, where $G_P = G_1$, $G_Q = G_2$, and $G_y = A$ are the respective stabilizers of the vertices and edges.

Proof. Up to isomorphism, X is the following graph:

$$\mathrm{vert}(X) = (G/G_1) \coprod (G/G_2)$$
$$\mathrm{edge}(X) = (G/A) \coprod \overline{(G/A)}$$

with the inclusions $A \to G_1$ and $A \to G_2$ inducing the maps $o : G/A \to G/G_1$ and $t : G/A \to G/G_2$. Put $P = 1 \cdot G_1$, $Q = 1 \cdot G_2$, and $y = 1 \cdot G_A$. T is then a fundamental domain for the natural action of G on X. The preceding theorem gives that X is a tree. □

These theorems establish an equivalence between amalgams of two groups and groups acting on trees with a segment as a fundamental domain.

Example 3.5. To show that $SL(2,\mathbb{Z})$ is an amalgam of two groups, we need to

(1) find a tree with a segment as a fundamental domain upon which $SL(2,\mathbb{Z})$ acts without inversion
(2) compute the stabilizers of the vertices and the edge of the fundamental domain

Fortunately, $SL(2,\mathbb{Z})$ acts in a well-known way on the upper half plane via Möbius transformations. Let y be the circular arc consisting of the points $z = e^{i\theta}$ for $\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$. Let $P = o(y) = e^{\pi i/3}$ and $Q = t(y) = e^{\pi i/2} = i$. Define the graph X to be the union of the transforms of y by the action of G. Thus PQ is our desired fundamental domain.

It can be easily shown the action of G is without inversion and that X is, in fact, a tree. Theorem 3.1 implies that G is an amalgam of the stabilizers of P and Q over the stabilizer of y:

G_P: Computing the stabilizer of P as such $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot e^{\pi i/3} = e^{\pi i/3}$ yields a cyclic subgroup of $G = SL(2, \mathbb{Z})$ of order 6 generated by $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$. Thus $G_P = \mathbb{Z}/6\mathbb{Z}$.

G_Q: Computing the stabilizer of Q as such $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot i = i$ yields a cyclic subgroup of $G = SL(2, \mathbb{Z})$ of order 4 generated by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Thus $G_P = \mathbb{Z}/4\mathbb{Z}$.

G_y: Since $G_y = G_P \cap G_Q$, we can see that $G_y = \mathbb{Z}/2\mathbb{Z}$.

We are now able to express $SL(2, \mathbb{Z})$ as an amalgam of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ over $\mathbb{Z}/2\mathbb{Z}$:

$$SL(2, \mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$$