Some existence results on the exterior Dirichlet problem for the minimal hypersurface equation

Nedir do Espírito-Santo
UFRJ
Instituto de Matemática
Caixa Postal 68530
21941-909 Rio de Janeiro-RJ
BRAZIL
nedir@im.ufrj.br

Jaime Ripoll
UFRGS
Instituto de Matemática
Av. Bento Gonçalves 9500
91540-000 Porto Alegre-RS
BRAZIL
jaime.ripoll@ufrgs.br

Abstract

It is proved the existence of solutions to the exterior Dirichlet problem for the minimal hypersurface equation in complete non compact Riemannian manifolds either with negative sectional curvature and simply connected or with nonnegative Ricci curvature under a growth condition on the sectional cuvature.

1 Introduction

With the development of the Riemannian Geometry many PDE results proved in the Euclidean geometry for the class of geometric operators have been investigated in more general Riemannian manifolds. In the case of the Laplacian equation, S. T. Yau proved that Liouville's Theorem (an entire non constant solution of the Laplace equation in $\mathbb{R}^n$ is necessarily unbounded) is also true in a complete Riemannian manifold M^n with non-negative Ricci curvature and conjectured that if M is simply connected with sectional curvature satisfying $-\kappa_1^2 \leq K_M \leq -\kappa_2^2 < 0$, for some constants κ_1, κ_2, then there should exist bounded solutions of the Laplacian equation on M. This has been proved in the 80's by Anderson and Sullivan (see [23]). Actually, they proved that on the compactified manifold $\overline{M} := M \cup \partial_\infty M$ one can solve the Dirichlet problem for the Laplace equation for any given continuous data on $\partial_\infty M$.

In the case of the mean curvature PDE equation, H. Rosenberg [20] extended Bernstein's Theorem (the only entire solutions of the minimal surface equation in $\mathbb{R}^2$ are the affine functions) to a complete 2-dimensional manifold with nonnegative sectional curvature. In [8] the authors proved the existence of entire solutions of the minimal hypersurface equation with any given smooth data at the asymptotic boundary $\partial_\infty M$ of M if M is complete, simply connected, with sectional curvature satisfying $K_M \leq -k^2 < 0$ and such that isotropy subgroup of the isometry group of M at some point p of M acts transitively on the geodesic spheres centered at p. This result has been substantially improved in [21].

We here study the existence of solutions to the exterior Dirichlet problem for the mean curvature PDE equation in a complete non compact Riemannian manifold M, namely, the existence of solutions to the following Dirichlet problem

$$\begin{cases} \mathcal{M}\left(u\right) := \operatorname{div}\left(\frac{\operatorname{grad} u}{\sqrt{1+|\operatorname{grad} u|^2}}\right) = 0 \text{ in } \Omega, \ u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1}$$

where Ω is an unbounded domain (open and connected) in M such that $\partial\Omega$ is compact; div and grad are the divergence and gradient in M.

The exterior Dirichlet problem for the minimal surface equation was first studied by Nitsche (see [17]) in the case that $M = \mathbb{R}^2$ and with Ω being the complement of a convex bounded domain. Later, Nitsche's result was extended and generalized by many authors as [10], [14], [19], but still in the Euclidean space.

In this work we prove existence results of solutions of (1) in the cases that either M has nonnegative Ricci curvature or M is simply connected and with sectional curvature K satisfying $K \leq -k^2 < 0$ for some positive constant k.

In the case of negative curvature, we prove:

Theorem 1 *Assume that M is simply connected and the sectional curvature K_M of M satisfies $K_M \leq -k^2 < 0$ for some positive constant k. We require that Ω is a $C^{2,\alpha}$ domain of M satisfying the exterior sphere condition, namely,* *given $p \in \partial\Omega$, there is a geodesic sphere of M passing through p, tangent to $\partial\Omega$ at p which is the boundary of a geodesic ball containing $\partial\Omega$.*

Given any nonnegative real number s, there exists a bounded solution $u \in C^{2,\alpha}(\overline{\Omega})$ of (1) such that

$$\sup_{\partial\Omega} |\operatorname{grad} u| = s$$

and

$$\max_{\Omega} |u| \leq \frac{s\,(3s^2+7)}{2k\,(s^2+1)}.$$

We next consider the case that M has nonnegative Ricci curvature. By a *soul* in M we mean a compact totally geodesic submanifold S of M (not necessarily convex) such that M is diffeomorphic to the normal bundle of S. We prove:

Theorem 2 *Let M be a n-dimensional, complete noncompact Riemannian manifold with nonnegative Ricci curvature,* $\mathrm{Ric}_M \geq 0$, *admitting a soul S and satisfying, for some $c > 0$,*

$$K_M(x) \leq \frac{4c^2}{\left(1+4c^2\rho^2(x)\right)^2},\ x \in M,$$

where ρ is the distance in M to S,

$$\rho(x) = \inf\left\{d(x,y) \mid y \in S\right\},$$

$d =$ Riemannian distance in M, and $K_M(x)$ is the maximum of the sectional curvature of M on planes of T_xM containing $\operatorname{grad}\rho$.

Let Ω be an unbounded domain in M such that $\partial\Omega$ is compact and $S \subset M\backslash\overline{\Omega}$. Then, for any given nonnegative real number s, there exists a solution $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ of (1) such that

$$\lim_{x\to\partial\Omega} \sup |\operatorname{grad} u(x)| = \sup_{\Omega} |\operatorname{grad} u| = s. \tag{2}$$

If M has nonnegative sectional curvature the existence of the soul is guaranteed by the Soul Theorem of Cheeger and Gromoll [3].

The vanishing boundary data hypothesis of the exterior Dirichlet problem, at least in Theorem 2, can not be dispensed, since a result of N. Kutev and

F. Tomi ([14]) proves the existence, when $M=\mathbb{R}^2$, of continuous non zero boundary data with arbitrarily small C^0 norm for which the exterior Dirichlet problem does not have a solution.

The Dirichlet problem for the minimal and constant mean curvature equation in other spaces than the Euclidean one have also been studied in [9], [13], [16], [18], [22]. The geometry of the minimal surfaces in product spaces of the form $M\times\mathbb{R}$ has also been investigated in [15]. Considering the more general notion of Killing graphs the Dirichlet problem for the minimal and constant mean curvature PDE is being also studied in more general spaces (that includes the product spaces $M\times\mathbb{R}$) (see [1], [2], [4], [5], [6], [7]).

2 Some notations and preliminary results

Is is assumed that M is a complete Riemannian manifold. We shall make use later of the following computation. Let N be any given smooth compact submanifold of M and denote by ζ the distance to N, that is,

$$\zeta(x)=d(x,N)=\min\{d(x,y)\mid y\in N\}$$

where d is the Riemannian distance in M. Given a function $\varphi\in C^2(\mathbb{R})$, set $w=\varphi(\zeta)$. If ζ is differentiable in a neighborhood of $x\in M$, considering an orthonormal basis in T_xM containing $\operatorname{grad}\zeta(x)$, we obtain

$$\mathcal{M}(w)=\operatorname{div}\left(\frac{\operatorname{grad} w}{\sqrt{1+|\operatorname{grad} w|^2}}\right)=\operatorname{div}\left(\frac{\varphi'\operatorname{grad}\zeta}{\sqrt{1+(\varphi')^2}}\right),$$

and, after some computations

$$\mathcal{M}(w)(x)=\frac{(n-1)\varphi''(\zeta(x))+\varphi'(\zeta(x))\left(1+(\varphi'(\zeta(x)))^2\right)\Delta\zeta(x)}{(n-1)\left(1+(\varphi'(\zeta(x)))^2\right)^{3/2}}. \tag{3}$$

The following result follows from Lemma 6 of [5]:

Lemma 3 *Let Λ be a $C^{2,\alpha}$ bounded open subset of M and $u\in C^{2,\alpha}(\overline{\Lambda})$ a solution of $\mathcal{M}[u]=0$ in Λ. Assume that u is bounded in Λ and that $|\operatorname{grad} u|$ is bounded in $\Gamma=\partial\Lambda$. Then $|\operatorname{grad} u|$ is bounded in Λ by a constant that depends only on $\sup_\Lambda |u|$ and $\sup_\Gamma |\operatorname{grad} u|$.*

In the case that M has nonnegative Ricci curvature we have in fact a maximum principle for the gradient. This principle is fundamental for the proof of Theorem 2:

Lemma 4 *Assume that the Ricci curvature of M is nonnegative. Let Λ be a C^∞ bounded open subset of M. Then any solution $u \in C^\infty(\overline{\Lambda})$ of $\mathcal{M}(u) = 0$ in Λ satisfies the gradient maximum principle*

$$\max_{\Lambda} |\operatorname{grad} u| = \max_{\partial\Lambda} |\operatorname{grad} u| .$$

Proof: Let η be a unit normal vector field orthogonal to the graph G of u such that $\langle \eta, T\rangle \geq 0$, where T is the Killing vector field $T(p,r) = (0,1) \in T_{(p,r)}M \times \mathbb{R}$. Note that

$$\langle \eta, T\rangle = \frac{1}{\sqrt{1 + |\operatorname{grad} u|}}.$$

Note that $\operatorname{Ric}_{M\times\mathbb{R}} \geq 0$ since $\operatorname{Ric}_M \geq 0$. Then, from Proposition 1 of [11]

$$\Delta \langle \eta, T\rangle = -(\operatorname{Ric}_{M\times\mathbb{R}}(\eta) + \|B\|^2) \langle \eta, T\rangle \leq 0,$$

where $||B||$ is norm of the second fundamental form of G. The function $\langle \eta, T\rangle$ is then superharmonic on G so that

$$\min_{G} \langle \eta, T\rangle = \min_{\partial G} \langle \eta, T\rangle$$

and then

$$\max_{\Lambda} |\operatorname{grad} u| = \max_{\partial\Lambda} |\operatorname{grad} u| .$$

Notation: Under the hypothesis and notations of Theorem 1, we denote by γ the Riemannian distance in M to $\partial\Omega$ restrict to $\overline{\Omega}$, namely

$$\gamma(p) = \inf \{d(p,q) \mid q \in \partial\Omega\}, \ p \in \overline{\Omega},$$

and under the hypothesis and notations of Theorem 2, we denote by ρ the Riemannian distance in M to S restrict to $\overline{\Omega}$. Moreover, we set $F_r = \gamma^{-1}(r)$, $r \geq 0$ and denote by H_{F_r} the normalized mean curvature of F_r with respect to the unit vector field normal to F_r pointing to the bounded connected component of $M\backslash F_r$. Note that

$$\Delta\gamma(x) = (n-1)H_{F_{\gamma(x)}}(x), \tag{4}$$

$x \in M$.

Lemma 5 *If M is simply connected and with sectional curvature satisfying $K_M \leq -k^2 < 0$, $k > 0$, then*

$$\inf_{F_r} H_{F_r} \geq k \tag{5}$$

for any $r > 0$.

Proof: Let $r > 0$ and $p_0 \in M$ be given. Denote by γ_{p_0} the distance function to p_0. Comparing the Hessian of γ_{p_0} with the Hessian of the distance function γ_k to a fixed point in a n-dimensional simply connected space M_k of constant sectional curvature $-k^2$ (Theorem 1.1 of [23]) we obtain, for any $x \in M_k$ such that $\gamma_k(x) = r$, any unit vector $X \in T_x M_k$ orthogonal to $\operatorname{grad} \gamma_k(x)$ and any unit vector $Y \in T_p M$ orthogonal to $\operatorname{grad} \gamma_{p_0}(p)$, where p is any point in the geodesic sphere $S_r(p_0) := \gamma_{p_0}^{-1}(r)$ centered at p_0 with radius r,

$$\langle Y, \nabla_Y \operatorname{grad} \gamma_{p_0} \rangle = \operatorname{Hes}(\gamma_{p_0})(Y, Y) \geq \operatorname{Hes}(\gamma_k)(X, X) = \langle X, \nabla_X \operatorname{grad} \gamma_k \rangle .$$

This implies that

$$\inf_{r>0} H_{S_r(p_0)} \geq k \coth(k\gamma_k) \geq k,$$

where $H_{S_r(p_0)}$ is the mean curvature of $S_r(p_0)$ with respect to the inner unit normal vector field.

Now, let $p \in F_r$ be given. There is $q \in \partial\Omega$ and a minimizing geodesic $\alpha : [0, r] \to M$ such that $\alpha(0) = q$ and $\alpha(r) = p$. Let $S_l(p_0)$ be a geodesic sphere of M with some radius l and some center p_0 passing through q, which is tangent to $\partial\Omega$ at q and such that $\partial\Omega \subset B_l(p_0)$. Then $S_{l+r}(p_0)$ passes to p and, by Gauss Lemma, is tangent to F_r at p. Moreover, since the normal exponential map from $\partial\Omega$ to Ω is a diffeomorphism, the geodesic ball $B_{l+r}(p_0)$ contains F_r. Since F_r is an embedded smooth hypersurface of M, it follows from the tangency principle that

$$H_{F_r} \geq H_{S_{l+r}(p_0)} \geq k$$

and finishes the proof of the lemma.

3 Proof of the Theorem 1.

Given $s \geq 0$, looking for constants a, b and c such that $\varphi(r) = (ar + b) / (r + c)$ determines a subsolution $v_s(x) := \varphi(\gamma(x))$ satisfying $v_s|_{\partial\Omega} = 0$ and $|\operatorname{grad} v_s|_{\partial\Omega} = s$ we obtain

$$\varphi(r) = \frac{s\left(3s^2+7\right)r}{2k\left(s^2+1\right)r+3s^2+7}.$$

To verify directly that φ is a subsolution note that

$$\varphi'(r) = \frac{s\left(3s^2+7\right)^2}{\left(2kr+3s^2+2krs^2+7\right)^2} \geq 0$$

so that, from (3), (4) and (5), we have that v_s is a subsolution if

$$\varphi''(\gamma) + k\varphi'(\gamma)\left(1+\left(\varphi'(\gamma)\right)^2\right) \geq 0.$$

A straightforward calculation shows that the left side of the above inequality can be written as the product of $1/\left(2k\gamma+3s^2+2ks^2\gamma+7\right)^6$ times a quartic polynomial on γ with positive coefficients, proving that v_s is a subsolution for Q on Ω.

Note that if $x \in \partial\Omega$ then

$$v_s(x) = \varphi(\gamma(x)) = \varphi(0) = 0$$

and

$$|\operatorname{grad} v_s(x)| = |\varphi'(0)|\,|\operatorname{grad}\gamma(x)| = \varphi'(0) = s.$$

Moreover

$$\lim_{\gamma(x)\to\infty} v_s(x) = \lim_{r\to\infty}\varphi(r) = \frac{s\left(3s^2+7\right)}{2k\left(s^2+1\right)}.$$

Let $m \in \mathbb{N}$ be given, $m \geq 1$. Setting $\Omega_m = \Omega \cap \{x \in \Omega \mid \gamma(x) < m\}$, we prove the existence of a solution $w_m \in C^{2,\alpha}(\overline{\Omega_m})$ of $\mathcal{M} = 0$ in Ω_m with $w_m|_{\partial\Omega} = 0$ and such that $\sup_{\partial\Omega}|\operatorname{grad} w_m| = s$. To this end, set

$$T_m = \left\{ \begin{array}{c} t \geq 0 \mid \exists u_t \in C^{2,\alpha}(\overline{\Omega_m}) \text{ such that } \mathcal{M}\left(u_t\right) = 0, \\ u_t|_{\partial\Omega} = 0,\ u_t|_{\Gamma_m} = t,\ \sup_{\partial\Omega}|\operatorname{grad} u_t| \leq s \end{array} \right\}$$

where $\Gamma_m = \partial\Omega_m \backslash \partial\Omega$.

We have $T_m \neq \emptyset$, since $0 \in T_m$. We prove that if $t \in T_m$ then $t \leq v_s|_{\Gamma_m}$, where v_s is defined in (i). Given $\varepsilon > 0$, we first prove that $t < v_{s+\varepsilon}|_{\Gamma_m}$. By contradiction, assume that $t \geq v_{s+\varepsilon}|_{\Gamma_m}$. Since

$$|\operatorname{grad} v_{s+\varepsilon}|_{\partial\Omega} = \inf_{\partial\Omega}|\operatorname{grad} v_{s+\varepsilon}| = s+\varepsilon > \sup_{\partial\Omega}|\operatorname{grad} u_t|$$

there is a neighborhood U of $\partial\Omega$ in Ω such that $u_t(x) < v_{s+\varepsilon}(x)$ for all $x \in U \backslash \partial\Omega$. It follows that there exists a domain $U \subset V \subset \Omega_m$ such that $u_t|_{\partial V} = v_{s+\varepsilon}|_{\partial V}$, what is an absurd since $v_{s+\varepsilon}|_V$ is a subsolution of $\mathcal{M}$ and hence $v_{s+\varepsilon}|_V \leq u_t|_V$. Letting $\varepsilon \to 0$ we have $t \leq v_s|_{\Gamma_m}$.

It follows that T_m is bounded and we may set

$$t_m = \sup T_m < \infty.$$

We prove that $t_m \in T_m$.

We first prove that there is a constant C, not depending on m, such that if $t \in T_m$, then $\sup_{\partial\Omega_m} |\operatorname{grad} u_t| \leq C$. Assume that $t \in T_m$. By definition of T_m we have $\sup_{\partial\Omega} |\operatorname{grad} u_t| \leq s$.

Setting $z_m = v_s|_{\Gamma_m}$ we have, as proved above, $z_m > t$. Moreover, the function $w_s := v_s - (z_m - t)$ is a subsolution of $\mathcal{M}$ in Ω_m and

$$\begin{aligned} w_s|_{\partial\Omega} &= -\left(z_m - t\right) \leq 0 = u_t|_{\Omega} \\ w_s|_{\Gamma_m} &= t = u_t|_{\Gamma_m}. \end{aligned}$$

It follows that

$$\begin{aligned} w_s &\leq u_t \leq t \\ w_s|_{\Gamma_m} &= t = u_t|_{\Gamma_m} \end{aligned}$$

and we may conclude that

$$\begin{aligned} \sup_{\Gamma_m} |\operatorname{grad} u_t| &\leq \sup_{\Gamma_m} |\operatorname{grad} w_s| = \varphi'(m) \\ &= \frac{s\left(3s^2+7\right)^2}{\left(2km + 3s^2 + 2kms^2 + 7\right)^2} \leq s\left(3s^2+7\right)^2 =: C. \end{aligned}$$

Then

$$\sup_{\partial\Omega_m} |\operatorname{grad} u_t| \leq C \tag{6}$$

for all $t \in T_m$.

Consider now a sequence $\{z_j\} \subset T_m$ converging to t_m as j goes to infinity. For each j, there is a function $u_j \in C^{2,\alpha}(\overline{\Omega_m})$ such that $\mathcal{M}\left(u_j\right) = 0$, $u_j|_{\partial\Omega} = 0$ and $u_j|_{\Gamma_m} = z_j$. Since, by the maximum principle,

$$\sup_{\Omega_m} |u_j| \leq t_m$$

it follows from (6) and Lemma 3 that the sequence $\{u_j\}$ has the C^1 norm uniformly bounded in $\overline{\Omega_m}$. Since $\overline{\Omega_m}$ is a compact $C^{2,\alpha}$ domain, from elliptic PDE theory we have C^2 compactness of $\{u_j\}$ in $\overline{\Omega_m}$. Hence, there is a subsequence of $\{u_j\}$ that converges uniformly C^2 in $\overline{\Omega_m}$ to a solution $w_m \in C^2(\overline{\Omega_m})$ of $\mathcal{M} = 0$ in Ω_m. From PDE elliptic regularity $w_m \in C^{2,\alpha}(\overline{\Omega_m})$.

The function w_m is a solution of $\mathcal{M} = 0$ in Ω_m that satisfies $w_m|_{\partial\Omega} = 0$, $w_m|_{\Gamma_m} = t_m$ and $\sup_{\partial\Omega_m} |\operatorname{grad} w_m| \leq s$. It follows that $t_m \in T_m$, that is, $w_m = u_{t_m}$.

We now note that $\sup_{\partial\Omega} |\operatorname{grad} w_m| = s$. In fact: By contradiction, assume that $\sup_{\partial\Omega} |\operatorname{grad} w_m| < s$.

Consider a function $\phi \in C^{2,\alpha}(\overline{\Omega_m})$ such that $\phi|_{\partial\Omega} = 0$ and $\phi|_{\Gamma_m} = t_m$, set

$$C_0^{2,a}(\overline{\Omega_m}) = \left\{ \omega \in C^{2,\alpha}(\overline{\Omega_m}) \;\middle|\; \omega|_{\partial\Omega_m} = 0 \right\},$$

and define $T\colon\ [0,2] \times C_0^{2,\alpha}(\overline{\Omega_m}) \to C^{\alpha}(\overline{\Omega_m})$ by

$$T\,(l,\omega) = \mathcal{M}\,(\omega + l\phi)\,.$$

Then

$$T\,(1,\omega_m) = 0$$

where $\omega_m = w_m - \phi$. From elliptic PDE theory we have that the Fréchet derivative $\partial_2 T\,(1,\omega_m) = d\mathcal{M}_{w_m}$ is a linear homeomorphism so that, from the implicit function theorem, there exists a continuous function (on the $C^{2,\alpha}$ topology) $i : (1-\varepsilon, 1+\varepsilon) \to C_0^{2,\alpha}(\overline{\Omega_m})$, with $i(1) = \omega_m$ such that $T\,(l, i(l)) = 0$, $l \in (1-\varepsilon, 1+\varepsilon)$. Therefore, since $|\operatorname{grad} w_m|_{\partial\Omega} < s$ and $w_m = i(1) + \phi$, there exists $l \in (1, 1+\varepsilon)$ such that $\sup_{\partial\Omega} |i(l) + l\phi| < s$. Since

$$0 = T(l, i(l)) = \mathcal{M}\,(i(l) + l\phi)$$

$i(l) + l\phi = 0$ at $\partial\Omega$ and $i(l) + l\phi = lt_m$ at Γ_m, we have that $lt_m \in T_m$, contradiction since $lt_m > t_m = \sup T_m$. Hence, $\sup_{\partial\Omega} |\operatorname{grad} w_m| = s$.

Since the estimates

$$\max_{\partial\Omega_m} |\operatorname{grad} w_m| \leq s\left(3s^2 + 7\right)^2$$

$$\max_{\Omega_m} |w_m| \leq \frac{s\,(3s^2+7)}{2k\,(s^2+1)}$$

do not depend on m it follows from Lemma 4 that the sequence $\{w_m\}$ has uniform C^1 estimates on compacts of $\overline{\Omega}$ which implies the existence of a subsequence of $\{w_m\}$ converging uniformly $C^{2,\alpha}$ on compacts of $\overline{\Omega}$ to a solution $u_s \in C^{2,\alpha}(\overline{\Omega})$ of $\mathcal{M} = 0$ in Ω satisfying $u_s|_{\partial\Omega} = 0$ and

$$\sup_{\partial\Omega} |\operatorname{grad} u_s| = s.$$

This concludes the proof of the theorem.

4 Proof of Theorem 2

We first consider the case that Ω is a C^∞ domain. Let $s \geq 0$ be given. We assume $s > 0$ otherwise the theorem is trivial. Since the soul S of M is contained in $M \backslash \overline{\Omega}$, the function ρ is smooth on Ω.

Given $m > 0$, set $B_m = \{\rho \leq m\}$ and let m_0 be such that $\partial\Omega \subset B_m$ for all $m \geq m_0$. Given $m > m_0$, set $\Omega_m = \Omega \cap B_m$ and $S_m = \partial\Omega_m \backslash \partial\Omega$.

We will prove the existence of $m_1 > m_0$ such that, given $m \geq m_1$, there is a solution $u_m \in C^\infty(\overline{\Omega_m})$ of $\mathcal{M} = 0$ in Ω_m with $u_m|_{\partial\Omega} = 0$ and $\sup_{\partial\Omega} |\operatorname{grad} u_m| = s$. To this end, given $m > m_0$, set

$$T_m = \left\{ \begin{array}{c} t \geq 0 \mid \exists u_t \in C^\infty(\overline{\Omega_m}) \text{ such that } \mathcal{M}(u_t) = 0, \\ \sup_{\Omega_m} |\operatorname{grad} u_t| \leq s, \; u_t|_{\partial\Omega} = 0, \; u_t|_{S_m} = t. \end{array} \right\}$$

It is clear that T_m is bounded and the uniform bound for the C^1 norm of the solutions of $\mathcal{M} = 0$ corresponding to points of T_m imply that the supremum $t_m = \sup T_m$ is attained. Moreover, we may make use of the implicit function theorem, as in the previous theorem, to assert that the solution $u_m \in C^\infty(\overline{\Omega_m})$ of $\mathcal{M} = 0$ in Ω_m such that $u_m|_{S_m} = t_m$ satisfies $\sup_{\Omega_m} |\operatorname{grad} u_{t_m}| = s$.

The main part of the proof consists in proving that the maximum of $|\operatorname{grad} u_{t_m}|$ is assumed at $\partial\Omega$. To this end, we construct barriers from above and from below to u_m which gradient less than or equal to $s/2$ at S_m.

Since $w_m := t_m$ is a solution of $\mathcal{M} = 0$ in Ω_m and $u|_{\partial\Omega_m} \leq w_m|_{\partial\Omega_m}$ we have $u_m \leq w_m$ on Ω_m.

To construct a barrier from below for (1) we first consider the paraboloid P of $\mathbb{R}^{n+1}$ obtained by acting the rotational group of $\mathbb{R}^{n+1}$, that leaves fixed the x_{n+1}-axis, on the parabola $(t, 0, ..., 0, ct^2)$, $t \in \mathbb{R}$. The sectional curvature of P at $y = (y_1, y_2, ..., y_{n+1}) \in P$ with respect to any plane through T_yP that contains the tangent vector of the geodesic passing through y and the vertex of P is given by

$$K_P(y) = \frac{4c^2}{\left(1 + 4c^2 r^2(y)\right)^2}$$

where $r(y) = \sqrt{y_1^2 + ... + y_n^2}$.

Let $x \in M$ and $y \in P$ be such that $\rho(x) = d_P(y, 0)$, where d_P is the Riemannian distance in P and 0 is the origin of $\mathbb{R}^{n+1}$. Then

$$\rho(x) = \int_0^{r(y)} \sqrt{1 + 4c^2 t^2}\,dt \geq r(y),$$

and, from the hypothesis, it follows that

$$K_M(x) \leq \frac{4c^2}{\left(1 + 4c^2 \rho^2(x)\right)^2} \leq \frac{4c^2}{\left(1 + 4c^2 r^2(y)\right)^2} = K_P(y).$$

We then obtain, from the Hessian Comparison Theorem (Theorem 1.1 of [23]), that $\Delta\rho(x) \geq \Delta\rho_P(y)$, where $\rho_P(y) = d_P(y, 0)$. Therefore, if φ is such that $\varphi' \geq 0$ it follows from (3) that $v(x) = \varphi(\rho(x))$ is a subsolution of Q in Ω if

$$(n-1)\varphi''(\rho_P(y)) + \varphi'(\rho_P(y))\left(1 + \left(\varphi'(\rho_P(y))\right)^2\right)\Delta\rho_P(y) \geq 0, \qquad (7)$$

for all $y \in P$.

We have

$$\Delta\rho_P(y) = \frac{n-1}{r(y)\sqrt{1 + 4c^2 r^2(y)}}. \qquad (8)$$

Introducing the notation

$$\delta(r) = \int_0^r \sqrt{1 + 4c^2 t^2}\,dt$$

and $\psi(r) = \varphi(\delta(r))$ we have

$$\varphi'(\delta(r)) = \frac{\psi'(r)}{\delta'(r)},$$

$$\varphi''(\delta(r)) = \frac{1}{(\delta'(r))^3}\left[\delta'(r)\psi''(r) - \psi'(r)\delta''(r)\right].$$

Setting $r = r(y)$ we have that the inequality (7) holds if and only if

$$\frac{1}{(\delta'(r))^3}\left\{\begin{array}{l} (n-1)\left[\psi''(r)\delta'(r) - \psi'(r)\delta''(r)\right] \\ +\psi'(r)\left[(\delta'(r))^2 + (\psi'(r))^2\right]\Delta\rho_P \end{array}\right\} \geq 0. \tag{9}$$

Since $\delta'(r) = \sqrt{1+4c^2r^2}$, $\delta''(r) = 4c^2r/\sqrt{1+4c^2r^2}$, from (8) we have that (9) holds if, and only if,

$$(n-1)\left[\psi''(r)\sqrt{1+4c^2r^2} - \frac{4c^2r\psi'(r)}{\sqrt{1+4c^2r^2}}\right]$$
$$+\psi'(r)\left[1+4c^2r^2+(\psi'(r))^2\right](n-1)\frac{1}{r\sqrt{1+4c^2r^2}} \geq 0$$

that is

$$\frac{(n-1)}{r\sqrt{1+4c^2r^2}}[r\left(1+4c^2r^2\right)\psi''(r) - 4c^2r^2\psi'(r)$$
$$+\psi'(r)\left(1+4c^2r^2\right) + (\psi'(r))^3] \geq 0$$

or

$$r\left(1+4c^2r^2\right)\psi''(r) + \psi'(r) + (\psi'(r))^3 \geq 0.$$

We have that

$$\psi_a(r) = \psi(r) = \sqrt{a}\int_{\frac{\sqrt{a}}{2c\sqrt{4-a}}}^{r} \frac{\sqrt{4c^2t^2+1}}{\sqrt{4c^2t^2(4-a)-a}}dt$$

is a solution of

$$r\left(1+4c^2r^2\right)\psi''(r) + \psi'(r) + (\psi'(r))^3 = 0$$

for all $a \in [0,4)$ and

$$r \geq \frac{\sqrt{a}}{2c\sqrt{4-a}}.$$

Moreover, ψ_a satisfies

$$\psi_a\left(\frac{\sqrt{a}}{2c\sqrt{4-a}}\right) = 0 \tag{10}$$
$$\psi_a'\left(\frac{\sqrt{a}}{2c\sqrt{4-a}}\right) = +\infty.$$

Let r_0 be such that $\delta(r_0) = m_0$ and a_0 such that

$$\frac{\sqrt{a_0}}{2c\sqrt{4-a_0}} = r_0,$$

that is,

$$a_0 = \frac{16c^2 r_0^2}{1+4c^2 r_0^2}.$$

Set $\varphi(\delta(r)) = \psi_{a_0}(r)$. Given $m > m_0$, let r_m such that $\delta(r_m) = m$. Since

$$\varphi'(m) = \frac{\psi'_{a_0}(r_m)}{\delta'(r_m)} = \frac{\sqrt{a_0}}{\sqrt{4c^2 r_m^2 (4-a_0) - a_0}}$$

we have $\varphi'(m) \leq s/2$ if and only if

$$\frac{\sqrt{a_0}}{\sqrt{4c^2 r_m^2 (4-a_0) - a_0}} \leq \frac{s}{2} \Leftrightarrow r_m \geq \frac{\sqrt{a_0\left(1+\frac{4}{s^2}\right)}}{2c\sqrt{4-a_0}}.$$

Hence, if we choose $m_1 > m_0$ such that

$$r_{m_1} \geq \frac{\sqrt{a_0\left(1+\frac{4}{s^2}\right)}}{2c\sqrt{4-a_0}},$$

since r_m increases with m we have $\varphi'(m) \leq s/2$ for all $m \geq m_1$. It follows from (10) that, for all $m \geq m_1$, $v_m(x) := \varphi(\rho(x))$ is a subsolution of $\mathcal{M}$ on $\Omega \cap (B_m \backslash B_{m_0})$ such that

$$v_m(x) = 0$$
$$|\operatorname{grad} v_m(x)| = \infty$$

if

$$\rho(x) = m_0$$

and

$$|\operatorname{grad} v_m(x)| \leq \frac{s}{2} \tag{11}$$

if $\rho(x) = m$.

Clearly $v_m + b$ is a subsolution of $\mathcal{M}$ on $B_m \backslash B_{m_0}$ for any constant b and there is b_0 such that $v_m(x) + b_0 \leq u_m(x)$ for all $x \in \Omega \cap (B_m \backslash B_{m_0})$. Set

$$b_m = \max\left\{b \mid v_m(x) + b \leq u_m(x),\ \forall x \in B_m \backslash B_{m_0}\right\}.$$

Since $|\operatorname{grad} v_m(x)| = \infty$ for $x \in S_{m_0} = \partial B_{m_0}$ it follows from the maximum principle that $v_m(x) + b_m = t_m$ for $x \in S_m$ and $u_m \geq v_m$ on $B_m \backslash B_{m_0}$. Then

$$|\operatorname{grad} u_m|_{S_m} \leq \max \left\{ |\operatorname{grad} v_m|_{S_m}, |\operatorname{grad} w_m|_{S_m} \right\} = \max \left\{ \frac{s}{2}, 0 \right\} = \frac{s}{2}.$$

It follows by Lemma 4 that

$$\sup_{\partial\Omega} |\operatorname{grad} u_{t_m}| = s.$$

Now, letting $m \to \infty$, the uniform C^1 estimates of u_{t_m} on compacts of Ω implies the existence of a subsequence of $\{u_{t_m}\}$ converging uniformly on compacts of Ω to a solution $u_s \in C^\infty(\overline{\Omega})$ of $\mathcal{M} = 0$ in Ω satisfying $u_s|_{\partial\Omega} = 0$ and $\sup_{\partial\Omega} |\operatorname{grad} u_s| = s$. Note that for each m there is a point $q_m \in \partial\Omega$ such that $|\operatorname{grad} u_{t_m}(q_m)| = s$. Since $\partial\Omega$ is compact a subsequence of q_m converges to $q \in \partial\Omega$. It follows that $|\operatorname{grad} u_s(q)| = s$, so that u_s can not be identically zero.

The case that $\partial\Omega$ is simply a C^0 domain is solved by approximating $\partial\Omega$ by C^∞ domains, applying the previous result and using the uniform C^1 estimates to establish the existence of a subsequence converging to a solution u_s of (1) satisfying (2). This concludes with the proof of the theorem.

References

[1] Alias, L. ; Dajczer, M.: "*Normal geodesic graphs of constant mean curvature*", Journal of Differential Geometry, v. 75, p. 387-401, 2007.

[2] Alias, L.; Dajczer, M.; Rosenberg, H.: "*The Dirichlet problem for CMC surfaces in Heisenberg space*", Calculus of Variations and Partial Differential Equations, v. 30, p. 513-522, 2007.

[3] Cheeger, J.; Gromoll, D.: *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. 96: 413-43, (1972)

[4] Dajczer, M.; Hinojosa, P. ; Lira, J. H. de: "*Killing graphs with prescribed mean curvature*", Calculus of Variations and Partial Differential Equations, v. 33, p. 231-248, 2008

[5] Dajczer, M; Lira, J. H. de: "*Killing graphs with prescribed mean curvature and Riemannian submersions*". Annales de l'Institut Henri Poincaré - Analyse non linéaire 26 (2009), 763–775.

[6] Dajczer, M.; Lira, J. H. de: *"Helicoidal graphs with prescribed mean curvature"*, Proceedings of the American Mathematical Society, v. 137, p. 2441-2444, 2009.

[7] Dajczer, M.; Ripoll, J.: "*An extension of a Theorem of Serrin to graphs in warped products",* The Journal of Geometric Analysis, Vol 15, N 2, 193 - 205 (2005)

[8] do Espírito-Santo, N., Fornari, S. and Ripoll, J.:"*The Dirichlet problem for the minimal hypersurface equation in $M \times \mathbb{R}$ with prescribed asymptotic boundary* ". Journal de Mathématiques Pures et Apliquées **93** (2) (2010).

[9] Elbert, M. F.; Rosenberg, H.: "*Minimal graphs in $M \times \mathbb{R}$*", preprint

[10] Earp, R. S.; Toubiana, E.: *Some applications of the maximum principle to hypersurface theory in Euclidean and hyperbolic space, New approaches in Nonlinear Analysis*, T. M. Rassis, Hadronic Press, 1999, 183 - 202

[11] Fornari, S.; Ripoll, J.: *Killing fields, mean curvature, translation maps",* Illinois Journal of Mathematics, Vol 48, N. 4, p 1385 - 1402, 2004

[12] Gilbarg, D.; Trudinger, N. S.: "*Elliptic partial differential equations of second order*", Springer-Verlag, 1998

[13] Hauswirth, L.; Rosenberg, H.; Spruck, J.: "*Infinite boundary value problems for constant mean curvature graphs in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$*", preprint

[14] Kutev, K; Tomi, F.: "*Nonexistence and instability in the exterior Dirichlet problem for the minimal surface equation in the plane*", Pacific Journal of Mathematics, Vol 170, N. 2, 1995

[15] W. H. Meeks, H.Rosenberg: "*The theory of minimal surfaces in $M \times \mathbb{R}$*", Comment. Math. Helv. 80 (2005)

[16] L. Mazet, M. M. Rodríguez, H. Rosenberg: "*The Dirichlet problem for the minimal surface equation with possible infinite boundary data over domains in a Riemannian surface*", preprint

[17] Nitsche, J. C. C.: "*Lectures on minimal surfaces*", Vol I, Cambridge University, 1989

[18] Nelli, B.; Rosenberg, H.: "*Minimal surfaces in* $H \times \mathbb{R}$", Bull Braz Math Soc (33), 2002, 263 - 292

[19] Ripoll, J.: "*Some characterization, uniqueness and existence results for Euclidean graphs of constant mean curvature with planar boundary*", Pacific Journal of Mathematics, Vol 198, N. 1, 2001, 175 - 196

[20] H. Rosenberg: "*Minimal surfaces in* $M \times \mathbb{R}$", Illinois Jr of Math, **46,** 1177 - 1195 (2002)

[21] J. Ripoll, M. Telichevesky: *"Some existence results for the Dirichlet problem for the minimal equation with prescribed asymptotic boundary in a negatively curved complete Riemannian manifold"*, preprint

[22] Spruck, J.: "*Interior gradient estimates and existence theorems for constant mean curvature graphs in* $M \times \mathbb{R}$", preprint, http://www.math.jhu.edu/~js/grad2.pdf

[23] R. Schoen, Yau, S.-T.: "*Lectures on Differential Geometry*", Conference Proceedings and Lecture Notes in Geometry and Topology, Vol I, International Press