ON QUADRATIC DIFFERENTIALS AND TWISTED NORMAL MAPS OF SURFACES IN $\mathbb{S}^2 \times \mathbb{R}$ AND $\mathbb{H}^2 \times \mathbb{R}$

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Abstract. Given a Lie group G with a bi-invariant metric and a compact Lie subgroup K, Bittencourt and Ripoll used the homogeneous structure of quotient spaces to define a Gauss map $\mathcal{N} : M^n \to \mathbb{S}$ on any hypersupersurface $M^n \looparrowright G/K$, where $\mathbb{S}$ is the unit sphere of the Lie algebra of G. It is proved in [BR] that M^n having constant mean curvature (CMC) is equivalent to $\mathcal{N}$ being harmonic, a generalization of a Ruh-Vilms theorem for submanifolds in the Euclidean space. In particular, when $n = 2$, the induced quadratic differential $\mathcal{Q}_{\mathcal{N}} := (\mathcal{N}^* g)^{2,0}$ is holomorphic on CMC surfaces of G/K.

In this paper, we take $G/K = \mathbb{S}^2 \times \mathbb{R}$ and compare $\mathcal{Q}_{\mathcal{N}}$ with the Abresch-Rosenberg differential $\mathcal{Q}$, also holomorphic for CMC surfaces. It is proved that $\mathcal{Q} = \mathcal{Q}_{\mathcal{N}}$, after showing that $\mathcal{N}$ is the *twisted normal* given by (1.5) herein. Then we define the twisted normal for surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and prove that $\mathcal{Q} = \mathcal{Q}_{\mathcal{N}}$ as well. Within the unified model for the two product spaces, we compute the tension field of $\mathcal{N}$ and extend to surfaces in $\mathbb{H}^2 \times \mathbb{R}$ the equivalence between the CMC property and the harmonicity of $\mathcal{N}$.

Dedicated to Keti Tenenblat on her 65th birthday

1. Introduction

For an oriented immersed surface Σ in the Euclidean space $\mathbb{E}^3$ with Gauss map $N : \Sigma \to (\mathbb{S}^2, g)$, it is well known that the following alternatives are equivalent:

a) Σ has constant mean curvature (CMC)

b) The Hopf's quadratic differential is holomorphic

c) N is harmonic.

The equivalence between a) and b) was obtained by H. Hopf [H] who used it to prove his celebrated theorem asserting that a topological CMC sphere in the Euclidean space is a round sphere. The equivalence between a) and c) holds for submanifolds of arbitrary codimension in $\mathbb{E}^n$ and was proved by Ruh-Vilms (see [RV]). The equivalence between b) and c) follows logically.

If the surface Σ is immersed in an arbitrary 3-dimensional Riemannian manifold M^3, the Hopf (quadratic) differential $\mathcal{A}$ is defined likewise, in terms

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of the second fundamental form. However, $\mathcal{A}$ fails to be holomorphic for CMC surfaces in general spaces.

In [AR], U. Abresch and H. Rosenberg defined a quadratic differential form $\mathcal{Q}$ of a surface Σ immersed in $\mathbb{S}^2 \times \mathbb{R}$, respectively in $\mathbb{H}^2 \times \mathbb{R}$, and extended Hopf's theorem for CMC spheres in these ambient spaces. Precisely,

$$\mathcal{Q} = 2H\mathcal{A} - \mathcal{T}, \text{ resp. } \mathcal{Q} = 2H\mathcal{A} + \mathcal{T}, \tag{1.1}$$

where H is the mean curvature of the surface, $\mathcal{A}$ is the Hopf differential and $\mathcal{T} = (dh \otimes dh)^{2,0}$, with h standing for the height function (see [AR]). They prove that $\mathcal{Q}$ is holomorphic when the surface is CMC. In particular, $\mathcal{Q} \equiv 0$ holds if Σ is a CMC topological sphere; from this fact, they obtain that a CMC sphere is rotationally symmetric.

In [BR], J. Ripoll and F. Bittencourt considered a hypersurface M^n oriented by a unit normal N in a homogeneous space G/K and defined the *Gauss map* $\mathcal{N}$ by taking the horizontal lifting $\widetilde{N}$ of N followed by the right translation of $\widetilde{N}$ to the group identity,

$$\mathcal{N}([g]) = (dr_{g^{-1}})_g \widetilde{N}(g) \in T_eG, \ [g] = gK \in M^n. \tag{1.2}$$

This map $\mathcal{N}$ takes values on the unit sphere of the Lie algebra $\mathcal{G}$ and coincides with the usual Gauss map N of a hypersurface in the Euclidean space considered as an abelian Lie group ($K = \{\vec{0}\}$ in this case). It follows from Corollary 4.4 of [BR]:

Theorem 1.1. *Let G endowed with a bi-invariant metric and K be a compact Lie subgroup. Given an immersed orientable hypersurface in the homogeneous space G/K, its Gauss map*

$$\mathcal{N} : M^n \to (\mathbb{S}^{n+k}, g) \subset \mathcal{G}, \ k = dim\ K,$$

is harmonic if and only if M^n has constant mean curvature.

This theorem applies to the homogeneous space $\mathbb{S}^2 \times \mathbb{R} = G/K$, with $G = SO(3) \times \mathbb{R}$ and $K = SO(2) \times \{0\}$. Thus, any CMC surface $\Sigma \looparrowright \mathbb{S}^2 \times \mathbb{R}$ is in correspondence with the harmonic Gauss map from [BR],

$$\mathcal{N} : \Sigma \to (\mathbb{S}^3, g) \subset so(3) \times \mathbb{R}. \tag{1.3}$$

We recall that any harmonic map on a surface induces a holomorphic quadratic differential (see 10.5 of [EL]). In particular, if Σ is a CMC surface in $\mathbb{S}^2 \times \mathbb{R}$, then $\mathcal{N}$ induces the holomorphic quadratic differential

$$\mathcal{Q}_{\mathcal{N}} := (\mathcal{N}^* g)^{2,0} = 2g(\mathcal{N}_z, \mathcal{N}_z) dz^2. \tag{1.4}$$

As we know, the Abresch-Rosenberg differential $\mathcal{Q}$ is also holomorphic on a CMC surface. How are $\mathcal{Q}_{\mathcal{N}}$ and $\mathcal{Q}$ related?

In Proposition 3.1 of this paper we prove that $\mathcal{Q}_{\mathcal{N}} = \mathcal{Q}$ holds for any surface in $\mathbb{S}^2 \times \mathbb{R}$. For that, we first show in Proposition 2.3 that if the unit normal is decomposed as $N = (V, \nu)$, then the Gauss map defined by (1.2) in [BR] satisfies

$$\mathcal{N} = (JV, \nu), \tag{1.5}$$

where J is the operator acting on tangent planes of $\mathbb{S}^2$ as the clockwise $\pi/2$-rotation. For that reason, we pass to call $\mathcal{N}$ the *twisted normal map*.

The computations to prove Proposition 3.1 clearly suggests that a similar result should work for a surface in $\mathbb{H}^2 \times \mathbb{R}$, once we consider $\mathbb{H}^2 \subset \mathbb{L}^3$ and adopt (1.5) as definition of $\mathcal{N}$. Precisely, we decompose the unit normal of $\Sigma \looparrowright \mathbb{H}^2 \times \mathbb{R}$ as $N = (V, \nu)$ and define the *twisted normal* of the surface by

$$\mathcal{N} = (JV, \nu) : \Sigma \to (d\mathbb{S}_3, g) \subset \mathbb{L}^3 \times \mathbb{R}, \tag{1.6}$$

now taking values in the 3-dimensional de-Sitter space $d\mathbb{S}_3$.

In the spirit of [RV], we compute the tension field of $\mathcal{N}$ for surfaces in both spaces $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ (Theorem 3.4 herein) and then prove that the surface has CMC if and only if $\mathcal{N}$ is harmonic, all done in an unified model for product spaces.

Summarizing the above facts, in the new diagram

a$'$) Σ is a CMC surface in $\mathbb{S}^2 \times \mathbb{R}$ or in $\mathbb{H}^2 \times \mathbb{R}$

b$'$) The Abresch-Rosenberg differential is holomorphic

c$'$) $\mathcal{N}$ is harmonic,

the alternatives a$'$) and c$'$) are equivalent, and imply b$'$).

It is worth to mention that the assertion b$'$) may not imply a$'$). In [FM] Fernandéz and Mira proved that if the *normalized* Abresch-Rosenberg differential, given by $\mathcal{Q}/H,\ H \neq 0$, is holomorphic on a surface in $\mathbb{H}^2 \times \mathbb{R}$, then it has constant mean curvature, except for a certain family of rotational surfaces not CMC. The question about a surface in $\mathbb{S}^2 \times \mathbb{R}$ with holomorphic $\mathcal{Q}/H$ being CMC remained open in [FM].

Very recently, Araújo and Leite [AL] determined all surfaces in both product spaces with $\mathcal{Q}$ holomorphic. For the sake of curiosity, a surface in $\mathbb{S}^2 \times \mathbb{R}$ satisfying b') but not a') is congruent to a unique rotational surface; when the ambient space is $\mathbb{H}^2 \times \mathbb{R}$, the surface is congruent to a piece of a unique surface foliated either by horizontal equidistant curves or by horizontal concentric circles. The rotational surfaces in both spaces have singularities, while those foliated by equidistant curves are complete graphs over $\mathbb{H}^2$.

2. Preliminaries

Let $F : \Sigma \looparrowright (M^3, \langle\ ,\ \rangle)$ denote the immersion of an oriented surface into a 3-dimensional manifold endowed with a Riemannian metric $\langle\ ,\ \rangle$. It is standard to consider the conformal structure on Σ induced by the metric of the immersion. If $z = x + iy$ is a complex coordinate of Σ and usual derivatives are indicated by a lower index, then

$$\langle F_x, F_x\rangle = \langle F_y, F_y\rangle = E > 0,\ \langle F_x, F_y\rangle = 0,$$

$$\langle F_z, F_z\rangle = \langle F_{\bar z}, F_{\bar z}\rangle = 0,\ \langle F_z, F_{\bar z}\rangle = E/2, \quad 2F_z = F_x - iF_y. \tag{2.1}$$

Let us denote by N the unit normal vector field orienting Σ and by $\overline{\nabla}$ the Levi-Civitta connection in M^3. One has that

$$\overline{\nabla}_{F_z} F_{\bar{z}} = (EH/2)\ N. \tag{2.2}$$

$$-\overline{\nabla}_{F_z} N = HF_z + (\alpha/E)\, F_{\bar{z}} \tag{2.3}$$

$$\overline{\nabla}_{F_z} F_z = (E_z/E)\, F_z + (\alpha/2)\, N. \tag{2.4}$$

If $\ell dx^2 + 2mdxdy + ndy^2$ is the local expression of the second fundamental form of the immersion, then $H = (\ell + n)/2E$ is the mean curvature of the surface and the coefficient α of the *Hopf differential* satisfies

$$\alpha = (\ell - n - i2m)/2, \quad \mathcal{A} = \alpha\ dz \otimes dz, \tag{2.5}$$

We note that $\mathcal{A}$ is a (well-defined) quadratic differential on Σ.

We now consider the Riemannian products

$$\mathbb{S}^2 \times \mathbb{R} \subset \mathbb{E}^3 \times \mathbb{R}, \quad \mathbb{H}^2 \times \mathbb{R} \subset \mathbb{L}^3 \times \mathbb{R}, \tag{2.6}$$

where $\mathbb{L}^3$ is the Minkowski space consisting of $\mathbb{R}^3$ with the non-degenerate scalar product $\vec{x}.\vec{x} = x_1^2 + x_2^2 - x_3^2$ and $\mathbb{H}^2 = \{\vec{x} : \vec{x}.\vec{x} = -1, x_3 \geq 1\}$. We use the dot "$\cdot$" notation for the metrics on Σ, $\mathbb{S}^2$, $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{E}^3$, in all cases restrictions of the Euclidean metric in $\mathbb{E}^3 \times \mathbb{R}$. Likewise, for the Riemannian metrics on $\mathbb{H}^2$ and $\mathbb{H}^2 \times \mathbb{R}$, restrictions of the Lorentzian metric in $\mathbb{L}^3 \times \mathbb{R}$.

Let $\mathbb{M}^2(c)$ stand for the base manifold $\mathbb{S}^2$ or $\mathbb{H}^2$, according to $c = 1$ or $c = -1$, respectively. We decompose the immersion $F = (f, h)$ and the normal $N = (V, \nu)$, so that

$$f(z) \in \mathbb{M}^2(c),\ h(z) \in \mathbb{R}, \quad V(z) \in T_{f(z)}\mathbb{M}^2(c),\ \nu(z) \in \mathbb{R}. \tag{2.7}$$

If ∇ denotes the connection in $\mathbb{M}^2(c)$, then the Riemannian connection $\overline{\nabla}$ in $\mathbb{M}^2(c) \times \mathbb{R}$ is the product of ∇ by the usual derivative on $\mathbb{R}$. By its turn, ∇ is given by the tangential component of the usual derivative on $\mathbb{E}^3$ or $\mathbb{L}^3$, according to $c = 1$, or $c = -1$, respectively.

The Codazzi equations are resumed to

$$\alpha_{\bar{z}} = E(H_z + c\nu h_z). \tag{2.8}$$

When H is constant, it follows from (2.8) and (2.2) that $\mathcal{Q}$ is holomorphic, with $\mathcal{T}$ given by

$$\mathcal{T} = 2(h_z)^2 dz^2. \tag{2.9}$$

We next recall the definition of the Gauss map given in [BR].

Definition 2.1. Let G/K be a homogeneous manifold of *left* residual classes in K, where G is a Lie group endowed with a *bi-invariant* metric and K is a compact Lie subgroup of G. Given an orientable hypersurface in G/K oriented by the unit normal field N, the *Gauss map* $\mathcal{N}$ of this hypersurface is the right translation to the Lie algebra $\mathcal{G}$ of the horizontal lifting $\widetilde{N}$ of N.

Notice that it suffices to take G with a *right-invariant* metric to have $\mathcal{N}$ well-defined by (1.2), since $r_{(gk)^{-1}} \circ r_k = r_{g^{-1}}$, right-translations are isometries in G and the horizontal lifting $\widetilde{N}$ satisfies $\widetilde{N}(gk) = (dr_k)_g(\widetilde{N}(g))$.

Let us consider the identification

$$\mathbb{S}^2 \times \mathbb{R} \equiv G/K, \quad G = SO(3) \times \mathbb{R}, \; K = SO(2) \times \{0\},$$

given by the map below, where A^j denotes the j^{th} column of $A \in SO(3)$:

$$(A,t)K \in G/K \mapsto (A^1, t) \in \mathbb{S}^2 \times \mathbb{R}. \tag{2.10}$$

Lemma 2.2. *Let $\Sigma \looparrowright \mathbb{S}^2 \times \mathbb{R}$, $A \in SO(3)$ and $t \in \mathbb{R}$ be given. Then:*

(i) the horizontal lifting of $N = (V, \nu)$ satisfies $\widetilde{N} = (\widetilde{V}, \nu)$, with $\widetilde{V}(A,t) = A \times Z$, for some $Z \perp so(2)$ in the Lie algebra $so(3)$.

(ii) Assume that Z has the form

$$Z = \begin{bmatrix} 0 & -a & -b \\ a & 0 & 0 \\ b & 0 & 0 \end{bmatrix} \in so(2)^{\perp} \subset so(3).$$

Under the identification (2.10), one obtains $\widetilde{V}(A^1, t) = aA^2 + bA^3 \in T_{A^1}\mathbb{S}^2$.

Proof. The first assertion follows from the linearity of multiplication by a fixed matrix.

To prove (ii), observe that the identification $ASO(2) \equiv A^1$ yields that $so(2)^{\perp}$ corresponds to $T_{A^1}\mathbb{S}^2$, spanned by the orthonormal basis $\{A^2, A^3\}$. Multiplication of matrices gives

$$A \times Z = [aA^2 + bA^3 \;\; -aA^1 \;\; -bA^1].$$

The expression of $\widetilde{V}(A^1, t)$ follows from (i). □

In the next result we prove that in $\mathbb{S}^2 \times \mathbb{R}$ the Gauss map $\mathcal{N}$ defined in [BR] is a twist of the standard normal of the surface.

Proposition 2.3. *Let $\Sigma \looparrowright \mathbb{S}^2 \times \mathbb{R}$ be oriented by the unit normal $N = (V, \nu)$. The map $\mathcal{N} : \Sigma \to \mathbb{S}^3 \subset so(3) \times \mathbb{R}$ satisfies $\mathcal{N} = (JV, \nu)$.*

Proof. It follows from definition and Lemma 2.2 that the $so(3)$-component of $\mathcal{N}$ is

$$\widetilde{V}(A,t) \times A^{-1} = A \times Z \times A^{-1}.$$

We obtain from multiplication of matrices that

$$A \times Z \times A^{-1} = \begin{bmatrix} 0 & -a\Delta_{33} + b\Delta_{32} & a\Delta_{23} - b\Delta_{22} \\ a\Delta_{33} - b\Delta_{32} & 0 & -a\Delta_{13} + b\Delta_{12} \\ -a\Delta_{23} + b\Delta_{22} & a\Delta_{13} - b\Delta_{12} & 0 \end{bmatrix},$$

where Δ_{ij} denotes the minor determinant of the entry A_{ij}.

Using that $A^{-1} = A^t \Leftrightarrow \Delta_{ij} = A_{ij}$ under the convention that

$$\begin{bmatrix} 0 & -t & s \\ t & 0 & -r \\ -s & r & 0 \end{bmatrix} \in so(3) \leftrightarrow (r,s,t) \in \mathbb{R}^3,$$

we have that the $\mathbb{E}^3$ component of $\mathcal{N}$ is $-bA^2 + aA^3$, thus $\mathcal{N} = (JV, \nu)$. □

3. Properties of the twisted normal map

In view of Theorem 1.1 and Proposition 2.3, we have that the twisted normal $\mathcal{N} = (JV, \nu) : \Sigma \to (\mathbb{S}^3, g)$ of a CMC surface in $\mathbb{S}^2 \times \mathbb{R}$ is harmonic, where $N = (V, \nu)$ is the unit normal orienting Σ.

A well-known property of harmonic maps imply that $\mathcal{Q}_{\mathcal{N}}$, defined in (1.3), is holomorphic. A surprising property is

Theorem 3.1. *$\mathcal{Q}_{\mathcal{N}} \equiv \mathcal{Q}$ holds for any oriented surface in $\mathbb{S}^2 \times \mathbb{R}$.*

Proof. It follows from $F_z = (f_z, h_z)$ that $0 = N.F_z = V.f_z + \nu h_z$. Moreover,

$$\mathcal{N}_z = d\mathcal{N}(f_z, h_z) = ((JV)_z, \nu_z)\,;\ (JV)_z = \nabla_{f_z}(JV) + [(JV)_z.f]f. \tag{3.1}$$

We point out that $\nabla_X(JY) = J(\nabla_X Y)$ holds for arbitrary vector fields, for J commutes with ∇ (to see that, take an orthonormal pair of parallel vector fields along the integral curve of X); in other words, every Riemann surface is Kähler. Also, $JV.f = V.f = 0$ implies that $(JV)_z.f = -JV.f_z$.

Now we replace the last equality in (3.1), using that J preserves the metric and also $f.f = 1$, to obtain that

$$\mathcal{N}_z.\mathcal{N}_z = \nabla_{f_z}(V).\nabla_{f_z}(V) + [JV.f_z]^2 + \nu_z^2. \tag{3.2}$$

Adding the first and third summands in (3.2):

$$\begin{aligned}\nabla_{f_z}(V).\nabla_{f_z}(V) + \nu_z^2 &= \overline{\nabla}_{(f_z,h_z)}(V,\nu).\overline{\nabla}_{(f_z,h_z)}(V,\nu) = \\ &\overline{\nabla}_{F_z}(N).\overline{\nabla}_{F_z}(N) = 2H(\alpha/E)F_z.F_{\bar z} = H\alpha;\end{aligned} \tag{3.3}$$

note that we have used (2.4) and then (2.1) in the second line of (3.3).

As for the second summand in (3.2), we claim that

$$[JV.f_z]^2 = -[h_z]^2. \tag{3.4}$$

Indeed,

$$[JV.f_z]^2 + [V.f_z]^2 = ||V||^2[f_z.f_z] \tag{3.5}$$

follows from the computations

$$\begin{aligned}&[JV.f_x]^2 - [JV.f_y]^2 - 2i[JV.f_x][JV.f_y] + [V.f_x]^2 - [V.f_y]^2 - 2i[V.f_x][V.f_y] = \\ &\qquad ||V||^2||f_x||^2 - ||V||^2||f_y||^2 - 2i||V||^2[f_x.f_y].\end{aligned}$$

Also, $[V.f_z]^2 = [-\nu h_z]^2$ and $||V||^2[f_z.f_z] = (1-\nu^2)(-[h_z]^2)$, so (3.4) holds.

Putting together (3.2), (3.3) and (3.4), we finally arrive at

$$2\mathcal{N}_z.\mathcal{N}_z = 2H\alpha - 2[h_z]^2. \tag{3.6}$$

Recalling (1.1) and (2.9), we conclude that $\mathcal{Q}_{\mathcal{N}} \equiv \mathcal{Q}$. □

Similar computations work for surfaces in $\mathbb{H}^2 \times \mathbb{R}$. Keeping the notations $F = (f, h)$ for the immersion and $N = (V, \nu)$ for the unit normal, one has that $f.f = -1$ and $N(p)$ lies on

$$d\mathbb{S}_3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 - x_3^2 + x_4^2 = 1\} \subset \mathbb{L}^3 \times \mathbb{R},$$

known as the *de-Sitter space*. Inspired by Proposition 2.3, we set

Definition 3.2. Given $\Sigma \looparrowright \mathbb{H}^2 \times \mathbb{R}$ oriented by the unit normal $N = (V, \nu)$, the *twisted normal map* of the surface is $\mathcal{N} = (JV, \nu) : \Sigma \to d\mathbb{S}_3$, where J is the complex structure on $\mathbb{H}^2$.

Theorem 3.3. *$\mathcal{Q}_{\mathcal{N}} \equiv \mathcal{Q}$ holds for any oriented surface in $\mathbb{H}^2 \times \mathbb{R}$.*

Proof. The same of Theorem 3.1, except for a few signals. Due to $f.f = -1$, the second equality in (3.1) becomes $(JV)_z = \nabla_{f_z}(JV) - \ [(JV)_z.f]f$ and the second summand of $\mathcal{N}_z.\mathcal{N}_z$ in (3.2) has a minus signal. Thus, equation (3.6) changes into $2\mathcal{N}_z.\mathcal{N}_z = 2H\alpha + 2[h_z]^2$ and $\mathcal{Q}_{\mathcal{N}} \equiv \mathcal{Q}$ holds. $\square$

Remark. If $G/K = \mathbb{E}^3$, then $\mathcal{Q}_{\mathcal{N}} = 2H\mathcal{A}$, with $\mathcal{N} = N$.

Theorem 3.4. *For any immersion $F : \Sigma \to \mathbb{M}^2(c) \times \mathbb{R}$, the tension field of the twisted normal map $\mathcal{N}$ is given by*

$$\widehat{\Delta}\mathcal{N} = -2\tilde{J} \circ dF(grad\ H) = (-2/E)\tilde{J}(H_x F_x + H_y F_y),$$

where $\tilde{J}(W, \beta) = (JW, \beta)$ and $\widehat{\nabla}$ stands for the connection in $\mathbb{S}^3$ or $d\mathbb{S}_3$, depending on $c = 1$ or $c = -1$, respectively.

Proof. Let us list some facts to be used in this proof:

(i) The *tension field of $\mathcal{N}$* is defined by $\widehat{\Delta}\mathcal{N} = tr(\widehat{\nabla} d\mathcal{N})$ (see [EL]). Its local expression is $(4/E)\widehat{\nabla}_{\mathcal{N}_{\bar{z}}}\mathcal{N}_z$.

(ii) $\widehat{\nabla}_{\mathcal{N}_{\bar{z}}}\mathcal{N}_z = \mathcal{N}_{z\bar{z}} - [\mathcal{N}_{z\bar{z}}.\mathcal{N}]\ \mathcal{N}$, for $\mathcal{N}.\mathcal{N} = 1$.

(iii) Let (W, β) be a tangent vector field along a curve $C = (f, h)$ in $\mathbb{M}^2(c) \times \mathbb{R}$ and $\dot{C} = (\dot{f}, \dot{h})$ be the curve velocity in $\mathbb{R}^4$. Then

$$(\dot{W}, \dot{\beta}) = \overline{\nabla}_{\dot{C}}(W, \beta) - c[W.\dot{f}](f, 0). \tag{3.7}$$

(iv) We have that $(JV)_z = \nabla_{f_z}(JV) + c\ [(JV)_z.f]f$, as in the proofs of Theorems 3.1 and 3.3. Also, (3.3) and (3.4) do not change. In particular, (3.4) implies that

$$JV.f_z = \pm i h_z,\ \ JV.f_{\bar{z}} = \mp i h_{\bar{z}}. \tag{3.8}$$

(v) $(\alpha/E)_{\bar{z}} + (\alpha/E^2)E_{\bar{z}} = (\alpha_{\bar{z}}/E) = H_z + c\nu h_z$, by Codazzi equation (2.8).

(vi) $\overline{\nabla}$ commutes with $\tilde{J}$ on $\mathbb{M}^2(c) \times \mathbb{R}$.

We now compute $\mathcal{N}_{z\bar{z}}$, from which $\widehat{\Delta}\mathcal{N}$ will be derived. It follows from $\mathcal{N} = \tilde{J}N$, (vi) and (3.7) that

$$-\mathcal{N}_z = \tilde{J}\left(-\overline{\nabla}_{F_z}N\right) + c[JV.f_z](f, 0).$$

Using that $\tilde{J}\left(\overline{\nabla}_{F_z}N\right) = (J(\nabla_{f_z}V), \nu_z)$, plus (vi) and (3.7), one has that

$$-\mathcal{N}_{z\bar{z}} = \overline{\nabla}_{F_{\bar{z}}}\tilde{J}\left(-\overline{\nabla}_{F_z}N\right) + c[J(\nabla_{f_z}V).f_{\bar{z}}](f, 0) + c\left\{[JV.f_z](f, 0)\right\}_{\bar{z}}. \tag{3.9}$$

It follows from (2.3) and (v) that the first summand satisfies

$$\overline{\nabla}_{F_{\bar{z}}}\tilde{J}\left(-\overline{\nabla}_{F_z}N\right) = \tilde{J}\ \overline{\nabla}_{F_{\bar{z}}}(HF_z + (\alpha/E)F_{\bar{z}}) =$$

$$H_{\bar{z}}\tilde{J}F_z + (H_z + c\nu h_z)\tilde{J}F_{\bar{z}} + \lambda\ \mathcal{N},\ \ \ \lambda = (EH^2/2) + (|\alpha|^2/2E). \tag{3.10}$$

As for the second summand of (3.9), the following holds:

$$J(\nabla_{f_z}V).f_{\bar{z}} = (\nabla_{f_z}JV).f_{\bar{z}} = [JV.f_{\bar{z}}]_z - JV.\nabla_{f_z}f_{\bar{z}} = [JV.f_{\bar{z}}]_z, \tag{3.11}$$

since (2.2) implies that $\nabla_{f_z} f_{\bar{z}}$ is parallel to V.

In view of (3.10) and (3.11), one obtains that

$$-\mathcal{N}_{z\bar{z}} = H_{\bar{z}}\tilde{J}F_z + (H_z + c\nu h_z)\tilde{J}F_{\bar{z}} + \lambda\,\mathcal{N} + 0(f,0) + c[JV.f_z](f_{\bar{z}},0); \quad (3.12)$$

the coefficient of $(f,0)$, given by $[JV.f_{\bar{z}}]_z + [JV.f_z]_{\bar{z}}$, vanishes due to (3.8).

We prove in the Appendix that the component of $[JV.f_z](f_{\bar{z}},0)$ orthogonal to $\mathcal{N}$ (either in $\mathbb{E}^4$ or $\mathbb{L}^4$) is precisely $-\nu h_z\tilde{J}F_{\bar{z}}$. Therefore, using (ii), we obtain that the tangential component of $-\mathcal{N}_{z\bar{z}}$ is just $H_{\bar{z}}\tilde{J}F_z + H_z\tilde{J}F_{\bar{z}}$. Since $\tilde{J}$ is linear and $2(H_{\bar{z}}F_z + H_zF_{\bar{z}}) = H_xF_x + H_yF_y = E[dF(grad\ H)]$, the theorem gets proved. □

Corollary 3.5. *Σ is a CMC surface in $\mathbb{H}^2\times\mathbb{R}$ if and only if the twisted normal map $\mathcal{N}:\Sigma\to(d\mathbb{S}_3,g)\subset\mathbb{L}^3\times\mathbb{R}$ is harmonic.*

Proof. It is immediate from Theorem 3.4 that $\mathcal{N}$ is harmonic if and only if H is constant. □

Remark 3.6. As in Corollary 3.5, we have an independent proof of Theorem 1.1 when $G/K = \mathbb{S}^2\times\mathbb{R}$.

4. Appendix

Let X denote the component of $(f_{\bar{z}},0)$ orthogonal to $\mathcal{N}=(JV,\nu)$. Since $(f_{\bar{z}},0).\mathcal{N} = f_{\bar{z}}.JV = \mp ih_{\bar{z}}$, we have that

$$X = (f_{\bar{z}},0) \pm ih_{\bar{z}}(JV,\nu) = (f_{\bar{z}} \pm ih_{\bar{z}}JV, \pm ih_{\bar{z}}\nu). \quad (4.1)$$

Let us observe that $f_{\bar{z}}$ never vanishes, since $f_x = f_y = 0$ at some point would imply that $E = h_x^2 = h_y^2 > 0$ and $h_xh_y = 0$, a contradiction.

We claim that $h_{\bar{z}}JV = \pm if_{\bar{z}} + \nu\ Jf_{\bar{z}}$ holds everywhere. Indeed, fix a point and write

$$h_{\bar{z}}JV = af_{\bar{z}} + b\ Jf_{\bar{z}},$$

with $a,b\in\mathbb{C}$. The claim gets proved from

$$h_{\bar{z}}JV.f_{\bar{z}} = a[f_{\bar{z}}.f_{\bar{z}}] \Longrightarrow h_{\bar{z}}(\mp i)h_{\bar{z}} = -a(h_{\bar{z}})^2 \Rightarrow a = \pm i,$$

$$h_{\bar{z}}JV.Jf_{\bar{z}} = b[f_{\bar{z}}.f_{\bar{z}}] \Longrightarrow -\nu(h_{\bar{z}})^2 = -b(h_{\bar{z}})^2 \Rightarrow b = \nu.$$

Therefore,

$$f_{\bar{z}} \pm ih_{\bar{z}}JV = f_{\bar{z}} \pm i(\pm if_{\bar{z}} + \nu\ Jf_{\bar{z}}) = \pm\ i\nu\ Jf_{\bar{z}} \Longrightarrow X = \pm\ i\nu(Jf_{\bar{z}}, h_{\bar{z}}).$$

Finally, the component of $[JV.f_z](f_{\bar{z}},0)$ orthogonal to $\mathcal{N}$:

$$\pm\ ih_zX = -h_z\nu(Jf_{\bar{z}},h_{\bar{z}}) = -\nu\ h_z\ \tilde{J}F_{\bar{z}}.$$

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