# 數學5342: 拓樸場論專題二

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作業2

Contents



The following exercises are taken from [1] and [2].

(1) Let $A$ be a differential graded algebra and $B, C$ be differential graded $A$-modules. The cobar resolution $\mathcal{B}_A(B, C)$ is given by

$$\mathrm{Hom}(B, C) \to \mathrm{Hom}(A \otimes B, C) \to \mathrm{Hom}(A \otimes A \otimes B, C) \to \cdots$$

where the map $d$ is given by

$$\begin{aligned} df(a_1 \otimes \cdots \otimes a_k \otimes b) = a_1 f(a_2 \otimes \cdots \otimes a_k \otimes b) \\ + \sum (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_k \otimes b) \\ + (-1)^k f(a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k b). \end{aligned}$$

Please verify that $d$ is a differential.

*Proof.* For the sake of clarity we will momentarily restrict our attention to the simpler case of $k = 3$, and from there extract the structure of the general argument (which ultimately amounts to an increase in the complexity of the notation involved). For the resent case, this result follows from a simple calculation:

$$\begin{aligned} d^2 f(a_1 \otimes a_2 \otimes a_3 \otimes b) &= a_1 df(a_2 \otimes a_3 \otimes b) - df(a_1 a_2 \otimes a_3 \otimes b) + df(a_1 \otimes a_2 a_3 \otimes b) - df(a_1 \otimes a_2 \otimes a_3 b) \\ &= a_1(a_2 f(a_3 \otimes b) - f(a_2 a_3 \otimes b) + f(a_2 \otimes a_3 b)) \\ &\quad - (a_1 a_2 f(a_3 \otimes b) - f(a_1 a_2 a_3 \otimes b) + f(a_1 a_2 \otimes a_3 b)) \\ &\quad + (a_1 f(a_2 a_3 \otimes b) - f(a_1 a_2 a_3 \otimes b) + f(a_1 \otimes a_2 a_3 b)) \\ &\quad - (a_1 f(a_2 \otimes a_3 b) - f(a_1 a_2 \otimes a_3 b) + f(a_1 \otimes a_2 a_3 b)) \\ &= (a_1 a_2 f(a_3 \otimes b) - a_1 a_2 f(a_3 \otimes b)) + (-a_1 f(a_2 a_3 \otimes b) + a_1 f(a_2 a_3 \otimes b)) \\ &\quad + (a_1 f(a_2 \otimes a_3 b) - a_1 f(a_2 \otimes a_3 b)) + (-f(a_1 a_2 a_3 \otimes b) + f(a_1 a_2 a_3 \otimes b)) \\ &\quad + (-f(a_1 a_2 \otimes a_3 b) + f(a_1 a_2 \otimes a_3 b)) + (f(a_1 \otimes a_2 a_3 b) - f(a_1 \otimes a_2 a_3 b)) \\ &= 0, \end{aligned}$$

as desired. The reader is encouraged to make sure that all of the terms are properly accounted for. From this, we can see that the terms cancel in pairs, and so in order to capture this feature for the general argument we will introduce some notation, and (hopefully) uncover the structural connection between these pairs.

The first application of $d$ yields $k + 1$ terms, which I will denote by $n_1, \cdots, n_{k+1}$. Applying $d$ once again yields $k$ terms for each of these $k + 1$ terms, which I will denote by $n_{ij}$, for $1 \le i \le k + 1$, $1 \le j \le k$ (e.g. $n_{24} = (dn_2)_4$ is the 4th term appearing in $dn_2$).

Then a brief glance at the above pairs reveals that the cancellation can be codified by following relation:

$$n_{ij} + n_{j+1,i} = 0 \quad \text{for } i = 1, \cdots, k;\ j = i, \cdots, k. \tag{$*$}$$

Notice that summing over all such terms yields

$$\sum_{i=0}^{k}\sum_{j=i}^{k}(n_{ij}+n_{j+1,i}) = \sum_{j\geq i} n_{ij} + \sum_{i=0}^{k}\sum_{j=i}^{K} n_{j+1,i} = \sum_{j\geq i} n_{ij} + \sum_{j<i} n_{ij} = \sum_{ij} n_{ij} = d^2 f(a_1 \otimes \cdots \otimes a_k \otimes b),$$

where the second equality comes from re-indexing the second summation ($j+1$ is replaced by $i$ and $i$ replaced by $j$); thus the validity of $(*)$ would establish the result.

In order to verify $(*)$, we will first introduce one more piece of notation to simplify the exposition. As it stands, the definition of $d$ involves 3 seemingly distinct terms, which will make the pairing of terms cumbersome and difficult to parse. As such, we will strip away all the unnecessary components of the notation, and focus on the underlying pattern, which will allow for us to write $d$ as a sum of terms of a single type; thereby making the pairing considerably easier to see. We let

$$(c, a_1, \cdots, a_k, b) := cf(a_1 \otimes \cdots \otimes a_k \otimes b).$$

Moreover, for simplicity, we will let $a_0 = 1$ and $a_{k+1} = b$. Then the definition of $d$ becomes

$$d(a_0, \cdots, a_{k+1}) = \sum_{i=0}^{k}(-1)^i(a_0, \cdots, a_i a_{i+1}, \cdots, a_{k+1}).$$

From here we can immediately right down: $n_i = (-1)^i(a_0, \cdots, a_i a_{i+1}, \cdots, a_{k+1})$ and $n_{j+1} = (-1)^{j+1}(a_0, \cdots, a_{j+1}a_{j+2}, \cdots, a_{k+1})$; hence

$$dn_i = (-1)^i\Bigg[\sum_{j<i}(-1)^j(a_0, \cdots, a_j a_{j+1}, \cdots, a_i a_{i+1}, \cdots, a_{k+1}) + \sum_{j\geq i}(-1)^j(a_0, \cdots, a_i a_{i+1}, \cdots, a_{j+1}a_{j+2}, \cdots, a_{k+1})\Bigg] \tag{$*'$}$$

and

$$dn_{j+1} = (-1)^{j+1}\Bigg[\sum_{i<j+1}(-1)^i(a_0, \cdots, a_i a_{i+1}, \cdots, a_{j+1}a_{j+2}, \cdots, a_{k+1}) + \sum_{i\geq j+1}(-1)^i(a_0, \cdots, a_{j+1}a_{j+2}, \cdots, a_i a_{i+1}, \cdots, a_{k+1})\Bigg] \tag{$*''$}$$

(where we split the sum into two pieces in order to account for the shift in the index when $j$ is larger than $i$ – since $a_i$ and $a_{i+1}$ now occupy the same position, the $(i+1)$-th spot is filled by $a_{i+2}$, and so on, so that the $j$-th spot is now occupied by $a_{j+1}$). Note that we are using the following convention

$$(a_0, \cdots, a_{j-1}a_j, a_j a_{j+1}, \cdots, a_{k+1}) = (a_0, \cdots, a_{j-1}a_j a_{j+1}, \cdots, a_{k+1})$$

in order to keep the notation simplified enough to allow for just two terms in both $(*')$ and $(*'')$.

Putting this all together, we can now say that given $j \geq i$ we have that $(dn_i)_j$ will come from the second sum in $(*')$, leaving us with

$$n_{ij} = (dn_i)_j = (-1)^{i+j}(a_0, \cdots, a_i a_{i+1}, \cdots, a_{j+1}a_{j+2}, \cdots, a_{k+1}).$$

Similarly, we see that $(dn_{j+1})_i$ will come from the first sum in $(*'')$, yielding

$$n_{j+1,i} = (dn_{j+1})_i = (-1)^{j+1+i}(a_0, \cdots, a_i a_{i+1}, \cdots, a_{j+1}a_{j+2}, \cdots, a_{k+1}).$$

These only differ by a power of -1, so we do indeed have that $n_{ij} + n_{j+1,i} = 0$. □

(2) We mentioned some axioms for $G$-equivariant open and closed TFTs: the $G$-twisted centrality condition, the $G$-twisted adjoint condition, and the $G$-twisted Cardy condition. Please draw the pictorial representations for these conditions. You can assume that there is only one label in the category of boundary conditions.

**Solution.**

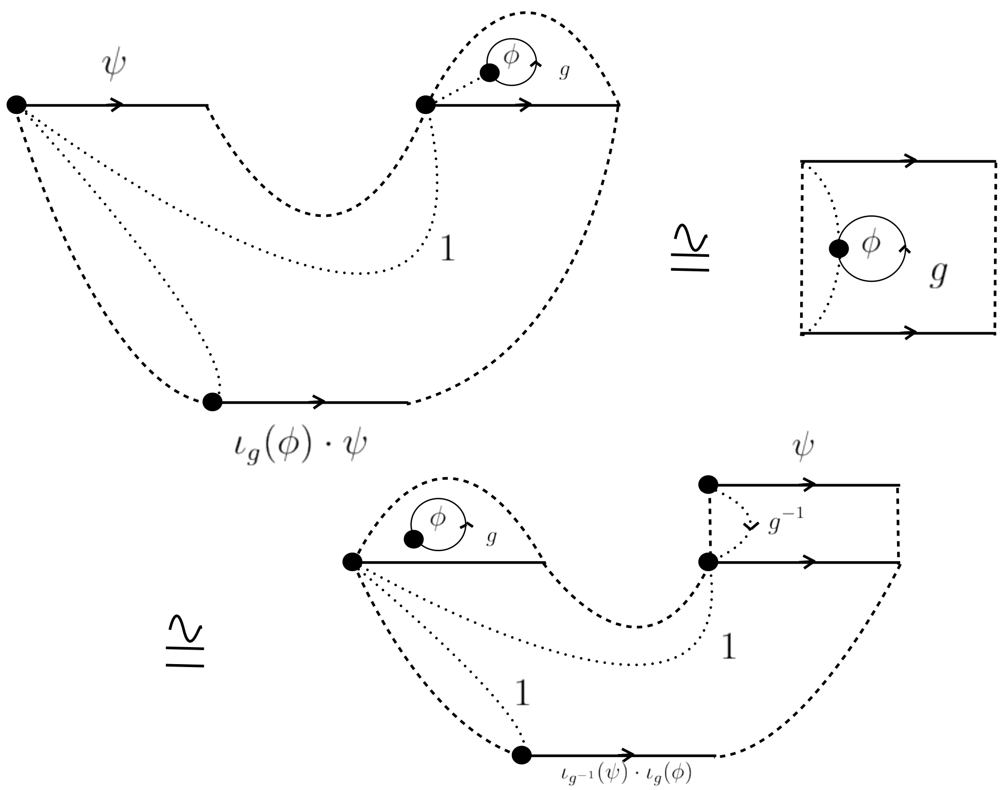


**Figure 1:** $G$-Twisted Centrality Condition

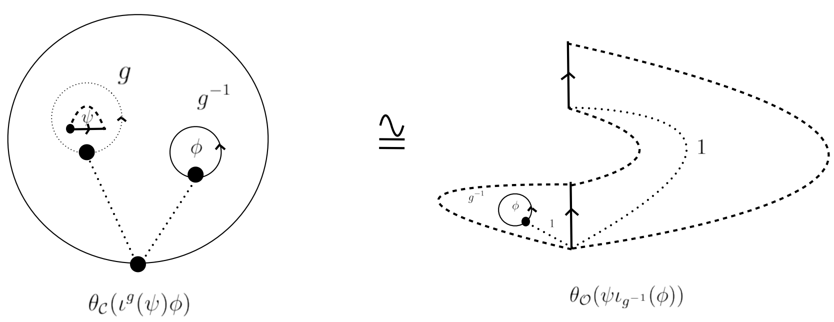


**Figure 2:** $G$-Twisted Adjoint Condition

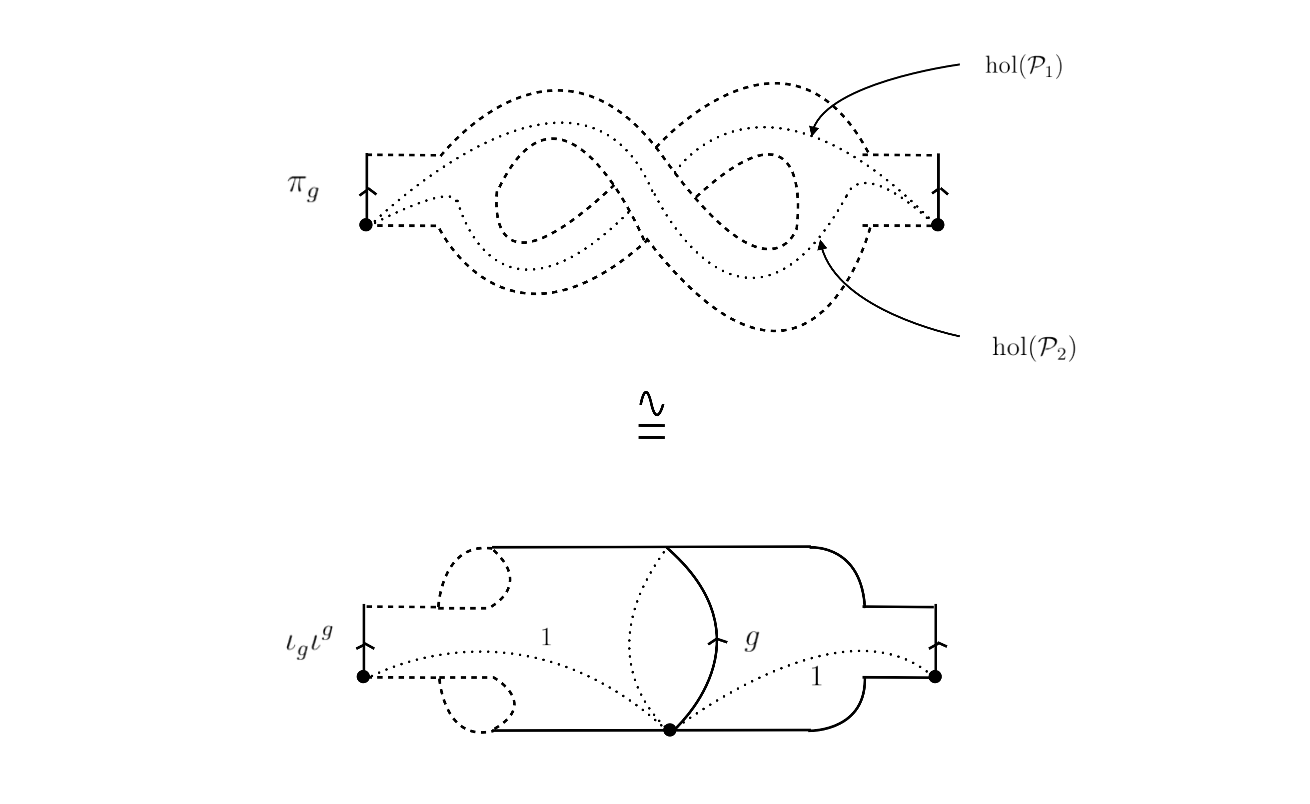


**Figure 3:** $G$-Twisted Cardy Condition

(3) (a) Write down an example of a special Lagrangian submanifold in $T^6$ and verify that it is indeed such a submanifold.

(b) Please explain why the topologically nontrivial (respectively, trivial) cycle in [1], Fig. 7 (page 196), has trivial (nontrivial) Maslov index.

**Solution.**

(a) As a subspace of $\mathbb{R}^6$, which is a 3-complex-dimensional manifold, $T^6$ inherits an anti-holomorphic involution as a subspace. Denote this involution by $\iota$ and define a submanifold $M$ as the fixed locus of $T^6$, i.e.

$$M := \{x \in T^6 \mid \iota(x) = x\}.$$

(Equivalently, one can say that this is the set of real points in $T^6$.) Then we claim that $M \subset T^6$ is a special Lagrangian submanifold. This follows from the following result:

**Theorem.** *Let $X$ be a Calabi-Yau manifold equipped with an anti-holomorphic involution $\iota : X \to X$. Then the fixed locus of $\iota$ is always a special Lagrangian submanifold.*

*Proof.* Let $\omega$ denote the Kähler form on $X$, and note that, in light of $X$ being a Calabi-Yau manifold, we have a unique holomorphic $n$-form $\Omega$. Since $\omega$ is a (1,1)-form, so too is $\iota^*\omega$; however, this implies that $\iota^*\omega$ is negative, or equivalently, that $-\iota^*\omega$ is positive. In particular, this implies that $-\iota^*\omega$ is Ricci-flat, and so the uniqueness of Ricci-flat metrics on Calabi-Yau manifolds thus implies that

$$-\iota^*\omega = \omega.$$

Therefore, if we restrict $\omega$ to the fixed locus $M$ of $\iota$, we have that it vanishes. Furthermore, this yields

$$\omega|_M \equiv 0 \qquad \Rightarrow \qquad \text{Re } \Omega|_M = 0 \quad \text{or} \quad \text{Im } \Omega|_M = 0.$$

In either case, $M$ is special Lagrangian, so we are done. □

Since $T^6$ has trivial first Chern class, it is a Calabi-Yau manifold, and so the an application of the above theorem establishes our claim.

(b) First, let us recall Figure 7 from [1]:

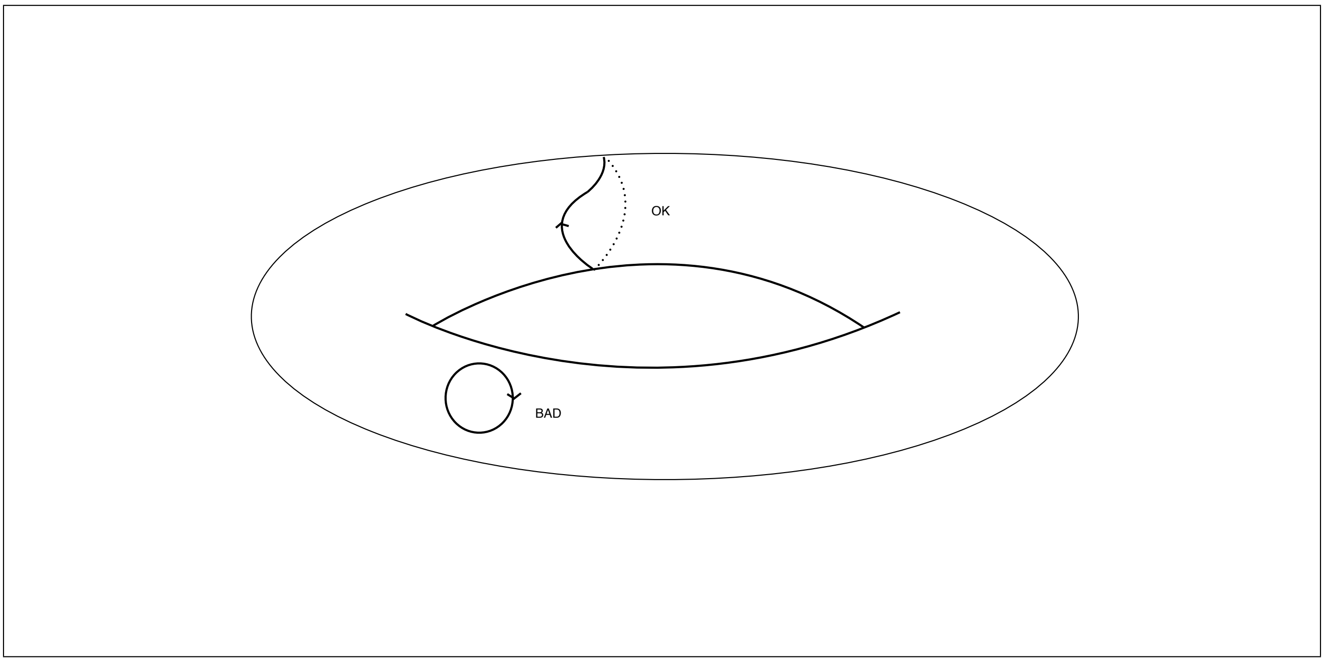


**Figure 7:** Loops which do and do not have trivial Maslov class

The Maslov class is a topological invariant which measures how much a Lagrangian submanifold's tangent space "turns" with respect to a given Lagrangian distribution on the ambient space (note that it does not depend on our choice of distribution, as it is a topological invariant). As such, one way of seeing this is to consider the Lagrangian distribution given by the the longitudinal curves on the torus. Clearly, as the curve labeled "good" (the nontrivial cycle) is one of the fibers in the foliation generating this distribution, it's tangent space does not "turn" with respect to the distribution, and so it's Maslov class is trivial, i.e. it constitutes a (potential) Lagrangian A-brane on the torus. As for the "bad" curve, it's tangent space is definitely turned with respect to this distribution, and so it will have a nontrivial Maslov class, i.e. it can be ruled out as a Lagrangian A-brane.

## References

[1] P. Aspinwall et al. *Dirichlet Branes and Mirror Symmetry*. Clay Mathematics Monographs, Vol. 4. American Mathematical Society, 2009.
[2] G. W. Moore and G. Segal. D-branes and K-theory in 2D Topological Field Theory. *ArXiv, hep-th/0609042*, September 2006.