數學5342：拓樸場論專題二

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作業1

CONTENTS

The following exercises are taken from [1] and [2].

(1) Suppose that the Frobenius algebra $\mathcal{C}$ associated with the closed string is semisimple, i.e. $\mathcal{C}$ is given by $\mathcal{C} = \oplus_x \mathbb{C}\epsilon_x$, where ϵ_x is an idempotent element satisfying $\epsilon_x \epsilon_y = \delta_{xy}\epsilon_x$.
   (a) Show that the most general $\mathcal{O}_{aa}$ is given by $\mathcal{O}_{aa} = \oplus_x \mathrm{End}(W_{x,a})$, for some family of vector spaces $W_{x,a}$. You are free to use [1], Theorem A and Theorem 2.6.
   (b) Show that $\mathcal{O}_{ab} \cong \oplus_x \mathrm{Hom}(W_{x,b}, W_{x,a})$. This is [1], Lemma 2.7. You may also use [1], Theorem A and Theorem 2.6 for this problem.
   (c) Use (b) combined with the axioms of 2D open and closed TFT (the adjoint relation and the Cardy condition, in particular) to deduce the following:

$$\theta_a(\psi) = \sum_x \sqrt{\theta_x}\mathrm{Tr}_{W_{x,a}}(\psi_x),$$
$$\iota^a(\psi) = \bigoplus_x \mathrm{Tr}_{W_{x,a}}(\psi_x)\frac{\epsilon_x}{\sqrt{\theta_x}},$$
$$\pi_b^a(\psi) = \bigoplus_x \frac{1}{\sqrt{\theta_x}}\mathrm{Tr}_{W_{x,a}}(\psi_x)P_{x,b}.$$

Solution.

(a) This follows almost immediately from Theorem 2.6 ([1]) and the Artin-Wedderburn theorem. The latter tells us that the most general $\mathcal{O}_{aa}$ is the direct sum of algebras, and the former says that each of these are given by the endomorphism ring of some vector space.

(b) We will break the proof up into three phases: the case with $a = b$, the case where $\mathcal{C}$ consists of a single idempotent, and finally the general case. The first case is almost trivial. From [1], Theorem 2.6, we immediately have, by definition, that

$$\mathcal{O}_{aa} \cong \bigoplus_x \mathrm{End}(W_{x,a}) = \bigoplus_x \mathrm{Hom}(W_{x,a}; W_{x,a}),$$

as desired. Moving on to the case when $\mathcal{C} = \mathbb{C}\varepsilon$, we begin by noting that since ι_a is central (in the sense of axiom 3c), we can see that $\iota_a(\varepsilon)\mathcal{O}_{ab}$ is actually a bimodule over both $\mathcal{O}_{aa}$ and $\mathcal{O}_{bb}$. Indeed, we have that

$$\iota_a(\varepsilon)\mathcal{O}_{ab} = \mathcal{O}_{ab}\iota_b(\varepsilon).$$

Moreover, a simple corollary to the Artin-Wedderburn theorem tells us that there is only one irreducible representation of $\mathcal{O}_{aa}$ (or, equivalently, only one irreducible module over $\mathcal{O}_{aa}$), which is given by W_a itself. Furthermore, the only $\mathcal{O}_{aa} \times \mathcal{O}_{bb}$-bimodule is given by $W_a^* \otimes W_b$. This means, of course, that we have that $\mathcal{O}_{ab} \cong$

$n_{ab}W_a^* \otimes W_b$ for some $n_{ab} \in \mathcal{N}$. This is almost what we need, as a classic result about vector spaces tells us that

$$W_a^* \otimes W_b \cong \mathrm{Hom}(W_a; W_b);$$

what remains to be shown is that $n_{ab} = 1$, so that we may apply this result. This restriction on n_{ab} arises as a consequence of the Cardy condition. Indeed, writing a basis for W_a as v_m and a basis for W_b as w_m, we have that

$$n_{ab}\mathrm{Tr}_{W_a}(\psi)P_b = \sum_{\mu}(w_{m,\alpha}^* \otimes v_{n,\alpha})\psi(v_{n,\alpha}^* \otimes w_{m,\alpha}) = \pi_b^a(\psi) = \iota_b \circ \iota^a(\psi) = \mathrm{Tr}_{W_a}(\psi)P_b;$$

thus $n_{ab} = 1$, and the result follows from what was said above. The latter phase is completely by merely applying what we have just proven to each piece of the direct sum over x indexing the idempotents.

(c) These follow from simple calculations. We will derive the second equality first. We begin by noting, as they did in [1], that $\theta_{\mathcal{C}}(\frac{\varepsilon_x}{\sqrt{\theta_x}}\frac{\varepsilon_y}{\sqrt{\theta_y}}) = \delta_{xy}$. Indeed,

$$\theta_{\mathcal{C}}\left(\frac{\varepsilon_x}{\sqrt{\theta_x}}\frac{\varepsilon_y}{\sqrt{\theta_y}}\right) = \frac{1}{\sqrt{\theta_x\theta_y}}\theta_{\mathcal{C}}(\varepsilon_x\varepsilon_y) = \frac{1}{\sqrt{\theta_x\theta_y}}\theta_{\mathcal{C}}(\delta_{xy}\varepsilon_y) = \frac{\delta_{xy}}{\sqrt{\theta_x\theta_y}}\theta_{\mathcal{C}}(\varepsilon_y) = \delta_{xy}\sqrt{\frac{\theta_y}{\theta_x}} = \delta_{xy}.$$

This means that $\varepsilon_x/\sqrt{\theta_x}$ form an orthonormal basis for $\mathcal{C}$. Thus $\iota^a(\oplus_x\psi_x) \in \mathcal{C}$ can be written as

$$\iota^a(\psi) = \bigoplus_x a_x(\psi_x)\frac{\varepsilon_x}{\sqrt{\theta_x}}$$

for some a_x depending on ψ_x – we simply need to show that this coefficient is given by the trace. To this end, we employ the adjoint relation in the following way:

$$\iota^a(\psi)\varepsilon_y = \bigoplus_x a_x(\psi_x)\frac{\varepsilon_x\varepsilon_y}{\sqrt{\theta_x}} = \bigoplus_x a_x(\psi_x)\frac{\delta_{xy}}{\sqrt{\theta_x}} = \frac{a_y(\psi_y)}{\sqrt{\theta_y}}$$

$$\Rightarrow \quad \frac{a_y(\psi_y)}{\sqrt{\theta_y}} = \theta_{\mathcal{C}}(\iota^a(\psi)\varepsilon_y) = \theta_a(\psi\iota_a(\varepsilon_y)).$$

Simplifying, we get the desired relation $a_x(\psi_x) = \mathrm{Tr}_{W_{x,a}}(\psi_x)$. The first equality also follows immediately from these considerations, just using $1_{\mathcal{C}}$ instead of ε_y and applying axiom 3d. The final equality follows immediately from the preceding two and the Cardy condition.

(2) Take the algebra $\mathcal{C} = H^*(X, \mathbb{C})$ with the trace map $\theta(\phi) = \int_X \phi$.

(a) Verify that the Frobenius algebra $\mathcal{O} = \mathcal{C} \otimes \mathrm{Mat}_N(\mathbb{C})$ with the trace map $\theta_{\mathcal{O}} = \int_X \mathrm{Tr}(\psi)$ does NOT give rise to an open and closed TFT. Which TFT condition is violated?

(b) (Optional) Let Y be a submanifold of X. Could you write down a version of open and closed TFT by using $\mathcal{O} = H^*(Y, \mathbb{C})\otimes\mathrm{Mat}_N(\mathbb{C})$ with the trace map $\theta_{\mathcal{O}} = \theta_0 \int_Y \mathrm{Tr}(\psi)$? Determine the value of the constant θ_0 and give some concrete examples.

Solution.

(a) The Cardy condition is the one that fails to be met. Indeed, we only have on choice for the maps ι^a and ι_a, which are

$$\iota_a(\phi) = \phi \otimes 1_N \quad \text{and} \quad \iota^a(\psi) = \mathrm{Tr}(\psi).$$

Thus

$$(\iota_b \circ \iota^a)(\psi) = \iota_b(\mathrm{Tr}(\psi)) = \mathrm{Tr}(\psi) \otimes 1_N.$$

On the other hand, however, we have that

$$\pi_b^a(\psi) = \sum(-1)^{\deg\omega_i(\deg\psi+\deg\omega^i)}\omega_i \otimes e_{lm} \wedge \psi \wedge \omega^i \otimes e_{ml} = \chi(TX) \wedge \mathrm{Tr}\psi,$$

where ω_i is a basis for H^*, e_{lm} are the units in $\mathrm{Mat}_N(\mathbb{C})$, and χ is the Euler class. Since this vanishes on forms of positive degree, it does not agree with what we derived above for the right side of the Cardy condition.

(b)

References

[1] P. Aspinwall et al. *Dirichlet Branes and Mirror Symmetry*. Clay Mathematics Monographs, Vol. 4. American Mathematical Society, 2009.

[2] G. W. Moore and G. Segal. D-branes and K-theory in 2D Topological Field Theory. *ArXiv, hep-th/0609042*, September 2006.