數學5345: 度量幾何

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作業1

Contents

(1) (Exercise 2.4.2, [1]) Give an example of a length structure on the plane $\mathbb{R}^2$ for which all continuous curves are admissible, the resulting intrinsic metric is the standard Euclidean one, but lengths of some curves differ from their Euclidean lengths.

Solution.

In order to ensure that the intrinsic metric remains unchanged, we must avoid altering any paths which realize the distance between two points – beyond that, we are quite free in modifying the length structure so that it is of the desired type. Such an example is furnished by the following: Let $\gamma_0 : [0,1] \to \mathbb{R}^2$ be the path in the plane tracing out the upper unit circle, i.e.

$$\gamma_0(t) = (\cos(\pi t), \sin(\pi t)).$$

Moreover, we define a family of paths A as

$$A := \{\gamma_0\} \cup \{\text{all restrictions of } \gamma_0 \text{ to subintervals of } [0,1]\}.$$

Let L denote the standard Euclidean length structure on the plane, and we will denote by L' the following new length structure:

$$L'(\gamma) = \begin{cases} L(\gamma), & \gamma \notin A \\ 2L(\gamma), & \gamma \in A \end{cases}$$

Using the fact that L is a length structure, L' can immediately be seen to be a length structure as well. Furthermore, we have not restricted the choice of curves at all, so all continuous curves are admissible, and the length of any curve $\gamma \in A$ will be, of course, different form the Euclidean length because it has been doubled. The intrinsic distance, however, remains the same in light of the fact that the distance between two point is realized by the straight line connecting two points and so it will not be affected by us enlarging some of the longer paths connecting those two points (like γ_0).

(2) (Exercise 2.3.11, [1], "Railway Metric") For two vectors $v, w \in \mathbb{R}^2$, set

$$d(v,w) = \Big| |v| - |w| \Big| + \min\{|v|, |w|\} \cdot \sqrt{\angle(v,w)},$$

where $\sqrt{\angle(v,w)}$ denotes the angle between v and w. Prove that (i) the topology determined by d is the standard Euclidean one.

(ii) The induced intrinsic metric $\hat{d}$ is

$$\hat{d} = \begin{cases} \Big||v| - |w|\Big|, & \text{if } \angle(v,w) = 0 \\ |v| + |w|, & \text{otherwise} \end{cases}$$

(iii) $(\mathbb{R}^2, \hat{d})$ is homeomorphic to the bouquet of a continuum of rays.

Proof. (i) Let d' denote the standard metric on the plane, and let

$$B_d(x,r) = \{y \in \mathbb{R}^2 \mid d(y,x) < r\},$$
$$B_{d'}(x,r) = \{y \in \mathbb{R}^2 \mid d'(y,x) < r\}.$$

The claim becomes almost trivial to prove after one simple observation:

$$d(\cdot, 0) = \Big||\cdot| - |0|\Big| + \min\{|\cdot|, |0|\} \cdot \sqrt{\angle(\cdot, 0)} = |\cdot| = d'(\cdot, 0);$$

thus $B_d(0,r) = B_{d'}(0,r)$. Since the latter forms a basis for the standard topology, so does the former, which immediately implies that the topologies are equivalent.

(ii) Since we are ultimately interested in the lengths of curves between any two points $v, w \in \mathbb{R}^2$, we will first set some notation for the sake of notational brevity: Let γ denote an arbitrary continuous path $\gamma : [0,1] \to \mathbb{R}^2$ (in light of the invariance of a length structure under reparameterization, considering $[0,1]$ will suffice for our purposes) such that $\gamma(0) = v$ and $\gamma(1) = w$. Moreover, let A be the set of all such curves.

From the generalized triangle inequality ([1], Prop. 2.3.4(i)), we have that for any $v, w \in \mathbb{R}^2$ and $\gamma \in A$

$$\begin{aligned} L_d(\gamma) &\geq d(\gamma(0), \gamma(1)) = d(v,w) \\ &= \Big||v| - |w|\Big| + \min\{|v|, |w|\} \cdot \sqrt{\angle(v,w)} \\ &\geq \Big||v| - |w|\Big|; \end{aligned}$$

thus $\Big||v| - |w|\Big|$ is an lower bound for $\{L_d(\gamma) \mid \gamma \in A\}$. In particular, this means that

$$\hat{d}(v,w) := \inf_{\gamma \in A}\{L_d(\gamma)\} \geq \Big||v| - |w|\Big|. \tag{$*$}$$

Restricting now to the case when $\angle(v,w) = 0$, we will (without loss of generality) say that $|v| \geq |w|$ and consider the particular path $\gamma_0(t) = tw + (1-t)v$. Then, by definition of the infimum, we see that

$$\begin{aligned} \hat{d}(v,w) &= \inf_{\gamma \in A}\{L_d(\gamma)\} \leq L_d(\gamma_0) \\ &= \sup_Y \left\{ \sum_{i=1}^{N} d(\gamma_0(y_{i-1}), \gamma_0(y_i)) \right\} \end{aligned}$$

where $Y = \{y_0, \cdots, y_N\}$ is a partition of $[0,1]$. Since $\angle(v,w) = 0$, they lie on a line through the origin, and consequently we have that $\angle(y_{i-1}, y_i) = 0$ for all $0 \leq i \leq N$. Using the definition of d, this simplifies the above to

$$\hat{d}(v,w) \leq \sup_Y \left\{ \sum_{i=1}^{N} \Big||\gamma_0(y_{i-1})| - |\gamma_0(y_i)|\Big| \right\};$$

however, γ_0 is simply traveling along the straight line through the origin from the outermost point of the two to the innermost point, so, for each i and any partition Y, $|\gamma_0(y_{i-1})| - |\gamma_0(y_i)|$ is merely the length of the line segment from $\gamma_0(y_{i-1})$ to $\gamma_0(y_i)$. Adding these up we obtain the usual notion of the length of γ_0 itself, which (since $\angle(v,w) = 0$) is given by $|v| - |w|$. This is independent of the partition, so we have established that $\hat{d}(v,w) \leq |v| - |w|$. Combining this with $(*)$ yields

$$\hat{d}(v,w) = \Big| |v| - |w| \Big|, \qquad \text{for } \angle(v,w) = 0.$$

As for the case when $\angle(v,w) \neq 0$, the fact that $\hat{d}$ is a metric (the triangle inequality, in particular) allows us to revert back to the case $\angle(v,w) = 0$ for half of the result:

$$\hat{d}(v,w) \leq \hat{d}(v,0) + \hat{d}(0,w) = \Big| |v| - |0| \Big| + \Big| |0| - |w| \Big| = |v| + |w|.$$

(iii) Clearly, $\mathbb{R}^2$ can be regarded as the wedge sum of a continuum of lines (the lines being the rays from the origin, and the point where they are all identified is, of course, the origin itself). As such, an explicit homeomorphism is not really required. Indeed, the distance in this space (between points with $\angle(v,w) \neq 0$ – the case when the angle vanishes is equally clear), defined by $\hat{d}$, is determined by traveling from the first point along a straight-line path back to the origin, and then going up another ray from the origin to the second point – precisely as one would do with the bouquet of rays.

□

(3) (Functional distance) Let (X,d) be a length space, and

$$d_{\text{functional}}(p,q) = \sup\{|f(p) - f(q)| \mid f : X \to \mathbb{R} \text{ is } 1-\text{Lipschitz function}\}$$

where the supremum is taken over all 1-Lipschitz functions defined on X.

(i) Show that $d_{\text{functional}}(p,q)$ is always less than or equal to the distance $d(p,q)$ determined by a length metric.

(ii) Show if X happens to be a complete smooth manifold, $d_{\text{functional}}(p,q)$ and $d(p,q)$ are equal if p and q are sufficiently close to each other, so then the functional distance is the same as the distance.

What about incomplete manifolds?

Proof. (i) Recall the following definition (adapted to our specific situation): A map $f : X \to \mathbb{R}$ is called *C-Lipschitz* if $|f(p) - f(q)| \leq Cd(p,q)$ for all $p,q \in X$. For the problem at hand, we therefore know that for any given $p,q \in X$ and 1-Lipschitz function f

$$|f(p) - f(q)| \leq d(p,q)$$

In particular, this reveals that $d(p,q)$ is an upper bound for the set $A := \{|f(p) - f(q)| \mid f \text{ is a } 1-\text{Lipschitz function}\}$. This implies that

$$d_{\text{functional}}(p,q) = \sup_{f \in A}\{|f(p) - f(q)|\} \leq d(p,q).$$

If we can show that the latter provides a lower bound on $L(\gamma)$ for any curve γ from p to q, then we will have

$$d(p,q) \leq \inf_{\gamma}\{L(\gamma)\} = \hat{d}(p,q),$$

from which the desired result follows when combined with the above inequality. We have already seen, however, how to do this (see the proof of (ii) in Exercise (2) above). We can apply the generalized triangle inequality ([1], Prop. 2.3.4(i)) to find that for any such γ

$$d_{\text{functional}}(p,q) \leq d(p,q) = d(\gamma(0),\gamma(1)) \leq L_d(\gamma).$$

Since nothing left of the second inequality depends on our choice of γ, we have established that it is, in fact, a lower bound, and we have

$$d_{\text{functional}}(p,q) \leq \hat{d}(p,q) \quad \text{for all } p,q \in X.$$

(ii) If X is assumed to be a smooth manifold, then it can be given a Riemannian metric (see, for example, Theorem 1.4.1 in [2]). Furthermore, the completeness condition implies that and two points $p,q \in X$ can be joined together by a geodesic γ of shortest length (given by $d(p,q)$, see: [2],Theorem 1.4.8). Thus

$$\hat{d}(p,q) = L_d(\gamma) = d(p,q).$$

For p and q sufficiently close, we can therefore make $\hat{d}(p,q) < \epsilon$ for any $\epsilon > 0$. From (i), however, we have that

$$d_{\text{functional}}(p,q) \leq \hat{d}(p,q),$$

so we can make this arbitrarily small as well. In particular, this means that we can ensure

$$|d_{\text{functional}}(p,q) - \hat{d}| \leq |d_{\text{functional}}(p,q)| + |\hat{d}| < \epsilon.$$

Since $d_{\text{functional}}$ is defined as a supremum, this means that $\hat{d}$ is not merely an upper bound, but is, in fact, the least upper bound, i.e.

$$d_{\text{functional}} = \hat{d},$$

as desired.

□

Since the above depended on the existence of geodesics realizing the distance between points, which is equivalent to the completeness condition according to the cited theorem, the same is not true of incomplete manifolds.

(4) Let (X,d) be a geodesic metric space and Y an arbitrary topological space with a distinguished class $\mathcal{F}$ of mappings $\{f : X \to Y\}$. Let

$$d_Y = \sup\left\{ d' \,\middle|\, d' \text{ is a metric on } Y, \text{ and } \sup_{x,y\in X}\left(\frac{d'(f(x),f(y))}{d(x,y)}\right) \leq 1 \text{ for all } f \in \mathcal{F} \right\}$$

denote the pointwise supremum of all metrics on Y for which, with respect to d', each function f in the class $\mathcal{F}$ is 1-Lipschitz.

(i) Show that d_Y is a (possibly degenerate) length metric on Y.
(ii) Give an example showing that d_Y can be degenerate (i.e. pseudometric in the sense that $d_Y(x,y) = 0$ for some points $x \neq y \in Y$.

Solution.

(i) *Proof.* □

(ii) It is not the case that d_Y can be degenerate. Indeed, given any metric d' on any space Y, by definition we *must* have that $d'(x,y) > 0$ for $x \neq y \in Y$. Therefore, without even worrying about the distinguished class of functions $\mathcal{F}$, or even the other space X, we can say that that for any collection A of metrics on Y we will have

$$d_Y(x,y) := \sup_{d'\in A}\{d'(x,y)\} > 0.$$

Moreover, the standard convention for a supremum over an empty set is $-\infty$, so even finding spaces for which there are no metrics satisfying the required condition would not give the desired result. The only thing I can think of would be to consider Y as a space with two points, and have d_Y defined as the *infimum* over such metrics.

(5) (Exercise 2.4.18, [1]) Prove that the completion of a length space is a length space.

Proof. Let (X, L) be a length space and let $\widetilde{X}$ be its completion, i.e. $X \subset \widetilde{X}$, $\widetilde{X}$ is complete, and $\overline{X} = \widetilde{X}$. Note that $\widetilde{X}$ is unique up to isometries that restrict to the identity on X. As such, we are free to consider the completion outlined in class (see 1.1 in the handout): $\widetilde{X} = \{\text{Cauchy seuquences in } X\}/\sim$, where $\sim$ is the equivalence relation defined by

$$\{x_n\} \sim \{y_n\} \quad \Leftrightarrow \quad \forall \epsilon > 0,\ \exists N = N(\epsilon) \text{ s.t. } \forall n > N,\ d(x_n, y_n) < \epsilon.$$

Moreover, we saw how one can extend the metric on X to $\widetilde{X}$ by defining

$$\widetilde{d}(\{x_n\}, \{y_n\}) = \lim_{n\to\infty} d(x_n, y_n).$$

Since $\widetilde{X}$ is therefore a metric space, it is also a length space. □

(6) (Exercise 2.5.24, [1]) Give an example of a complete length space (with finite order) which is not a geodesic space (i.e., for which there is no shortest path between some points).

Solution.

Consider the set $A \subset \mathbb{R}^2$ defined by

$$A := \bigcup_{n\in\mathbb{N}} \gamma_n([0,1]),$$

where $\gamma_n : [0,1] \to \mathbb{R}^2$ is a (continuous) path with length $1 + \frac{1}{n}$ such that $\gamma(0) = (0,0)$ and $\gamma(1) = (1,0)$. The fact that it is a length space is immediately clear and equally clear is the fact that A is not geodesic, as, for example, the points $(0,0)$ and $(1,0)$ cannot be connected by a geodesic (the only geodesic between them is given by the straight line segment connecting them, which is not included in the graphs of the γ_n); however, it is a complete length space. Indeed, it is a closed subspace of a complete metric space, and is therefore complete.

References

[1] D. Burago, Y. Burago, and S. Ivanov. *A Course in Metric Geometry*. Graduate Studies in Mathematics, Vol. 33. American Mathematical Society, 2001. http://www.math.psu.edu/petrunin/papers/alexandrov/bbi.pdf.

[2] J. Jost. *Riemannian Geometry and Geometric Analysis*. Springer, 2002.